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PROPOSITION 1. *Let  $M$  be a complex surface, and assume that its fundamental group  $G$  fulfills  $p(G) \geq 0$ . Then the holomorphic Euler characteristic of  $M$  is  $\geq 0$ .*

By the Kodaira-Enriques classification it follows that  $M$  cannot be ruled over a curve of genus  $\geq 2$ .

REMARK. The formulae above leading to the holomorphic Euler characteristic refer to the orientation of the complex surface dictated by the complex structure. Thus the argument is valid only if in *that* orientation  $\sigma(M) \leq 0$ . If however  $\sigma(M) > 0$  then  $p(G) \geq 0$  implies that  $2 - 2\beta_1(G) + 2\beta_2^+_{\text{wrong}}(M) \geq 0$  where  $\beta_2^+_{\text{wrong}}$  refers to the “wrong” orientation and is  $= \beta_2^-(M)$ . Now  $\beta_2^+(M) > \beta_2^-(M)$  by assumption. Thus the result remains true; the holomorphic characteristic is  $> 0$ .

III) *Donaldson Theory*. Finitely presented groups  $G$  with  $p(G) \geq 0$  and  $\beta_1(G) \geq 4$  do not qualify for the Theorems A, B, and C of Donaldson [D] relating to non-simply connected topological manifolds. Indeed in these theorems the signature is assumed to be negative with  $\beta_2^+ = 0, 1$  or  $2$ . However  $p(G) \geq 0$  means  $2 - 2\beta_1(G) + 2\beta_2^+(M) \geq 0$ , i.e.  $\beta_2^+(M) \geq \beta_1(G) - 1$ .

#### 4. DEUS EX MACHINA: $l_2$ -COHOMOLOGY

4.1. We recall in a few words the (cellular) definition of  $l_2$ -cohomology and  $l_2$ -Betti numbers, in the case of a 4-manifold  $M$  but things apply to any finite cell-complex.

Some definitions: For any countable group  $G$  let  $l_2G$  be the Hilbert space of square-integrable real functions on  $G$ , with  $G$  operating on the left, and  $NG$  the algebra of bounded  $G$ -equivariant linear operators on  $l_2G$ . A Hilbert- $G$ -module  $H$  is a Hilbert space with isometric left  $G$ -action which admits an isometric  $G$ -equivariant imbedding into some  $l_2G^m$  (direct sum of  $m$  copies of  $l_2G$ ). The projection operator  $\phi$  of  $l_2G^m$  with image  $H$  is given by a matrix  $(\phi_{kl})$ ,  $\phi_{kl} \in NG$ . The “trace”  $\sum \langle \phi_{kk}(1), 1 \rangle$  is the von Neumann dimension  $\dim_G H$ ; it is a real number  $\geq 0$ , and  $= 0$  if and only if  $H = 0$ .

Let  $\tilde{M}$  be the universal cover of  $M$  with the cell-decomposition corresponding to that chosen in  $M$ . The square-integrable real  $i$ -cochains of  $\tilde{M}$  constitute a Hilbert space  $C_{(2)}^i(\tilde{M})$  with isometric  $G$ -action. It decomposes into the direct sum of  $\alpha_i$  copies of  $l_2G$ ,  $i = 0, \dots, 4$ . As before  $\alpha_i$  denotes the

number of  $i$ -cells of  $M$ ;  $G$  is the fundamental group of  $M$  acting by permutation of the cells of  $\tilde{M}$ . The  $C_{(2)}^i$  with the induced coboundary operators form a Hilbert- $G$ -module chain complex. The cohomology  $H^i$  of that complex is easily identified with  $H^i(M; l_2 G)$ , cohomology with local coefficients (see, e.g. [E2]). The *reduced* cohomology group  $\bar{H}^i$  (i.e. cocycles modulo the closure of coboundaries) of that complex can be imbedded in  $C_{(2)}^i$  as a  $G$ -invariant subspace and is therefore a Hilbert- $G$ -module. Its von Neumann dimension  $\dim_G \bar{H}^i(\tilde{M})$  is the  $i$ -th  $l_2$ -Betti number  $\bar{\beta}_i(M)$ . It is a topological, even a homotopy, invariant of  $M$ .

4.2. Since  $\dim_G C_{(2)}^i = \alpha_i$  and since the von Neumann dimension behaves like a rank, the usual Euler-Poincaré argument shows that the  $l_2$ -Betti numbers compute the Euler characteristic exactly as the ordinary Betti numbers do:

$$\chi(M) = \sum (-1)^i \bar{\beta}_i(M).$$

Moreover the  $\bar{\beta}_i$  of a closed manifold fulfill Poincaré duality. Thus

$$\chi(M) = 2\bar{\beta}_0 - 2\bar{\beta}_1 + \bar{\beta}_2.$$

According to Atiyah's  $l_2$ -signature theorem [A],  $\sigma(M)$  can also be expressed by appropriate  $l_2$ -Betti numbers:  $\bar{H}^2(\tilde{M})$  splits into two complementary  $G$ -invariant subspaces with von Neumann dimensions  $\bar{\beta}_2^+(M)$  and  $\bar{\beta}_2^-(M)$ , and  $\sigma(M)$  is their difference. Thus, as with ordinary Betti numbers, one has

$$\chi(M) + \sigma(M) = 2\bar{\beta}_0(G) - 2\bar{\beta}_1(G) + 2\bar{\beta}_2^+(M).$$

We now assume  $G$  to be infinite. Then  $\bar{\beta}_0(G) = 0$ . Indeed a 0-cocycle  $f$  in  $\tilde{M}$  is a constant and if  $\tilde{M}$  is an infinite complex  $f$  can be  $l_2$  only if it is  $= 0$ .

**THEOREM 2.** *If for a finitely presented group  $G$  the first  $l_2$ -Betti number  $\bar{\beta}_1(G)$  is 0 then the invariants  $p(G)$  and  $q(G)$  are non-negative.*

**COROLLARY 3.** *If  $\bar{\beta}_1(G) = 0$  then  $\text{def}(G) \leq 1$ .*

**COROLLARY 4.** *If  $G = \pi_1(\text{complex surface } M)$  with  $\bar{\beta}_1(G) = 0$  then the holomorphic Euler characteristic of  $M$  is non-negative.*

4.3. There are many groups for which it is known that  $\bar{\beta}_1(G) = 0$ . A good list is given in [B-V]. We mention here three big and interesting classes of groups with that property.

1) All finitely generated amenable groups [C-G]. We recall that this class includes the virtually solvable groups, thus in particular the finitely generated Abelian groups (whence  $\mathbf{Z}^n$ , example 1) in 2.2). [Actually for an amenable group  $G$  with  $K(G, 1)$  of finite type, i.e. there is a  $K(G, 1)$  with finite  $m$ -skeleta, all  $l_2$ -Betti numbers are 0.]

THEOREM 5. *If  $G$  is a finitely presented amenable group then  $p(G)$  and  $q(G)$  are non-negative.*

2) [L1] All finitely presented groups  $G$  containing an infinite finitely generated normal subgroup  $N$  such that there is in  $G/N$  an element of infinite order. For these “Lück groups” one has the same conclusions as in the amenable case. — In [L1] the subgroup  $N$  is assumed to be finitely presented. Lück has shown later [L2] that the weaker assumption above is sufficient.

3) The statement of Theorem 5 also holds more generally for a finitely presented group  $G$  which contains a finitely generated normal subgroup  $N$  such that  $G/N$  is infinite and amenable [E2]. The proof is somewhat different: It makes use not of the universal cover but of the cover belonging to  $N$ . The amenable group  $G/N$  operates on that cover and one can use the  $l_2$ -Betti numbers relative to  $G/N$ . — A simple example is given by a group with finitely generated commutator subgroup and infinite Abelianisation.

#### 4.4. REMARKS.

1) We note that for finitely presented infinite amenable groups, and also for groups as in 4.3, 3) above, the deficiency is  $\leq 1$ . This can also be proved without 4-manifolds: It suffices to consider a  $K(G, 1)$  with 2-skeleton corresponding to a presentation of  $G$ .

2) It is well-known that a group with deficiency  $\geq 2$  cannot be amenable since it contains free subgroups of rank  $\geq 2$ ; see [B-P], where a stronger result is proved.

3) There is a class of groups for which  $\overline{\beta}_1$  is positive: The groups  $G$  with infinitely many ends (i.e. with  $H^1(G; \mathbf{Z}G)$  of infinite rank; here one takes ordinary cohomology with local coefficients). A nice proof for this can be found in [B-V]. Another approach is to use Stallings’ structure theorem from which it follows that these groups contain free subgroups of rank  $\geq 2$  and thus are non-amenable. For non-amenable groups the Guichardet amenability criterion [G] tells that  $\overline{H}^1(G; l_2 G) = H^1(G; l_2 G)$ . The coefficient

map  $H^1(G; \mathbf{Z}G) \longrightarrow H^1(G; l_2 G)$  induced by the imbedding  $\mathbf{Z}G \longrightarrow l_2 G$  is easily seen to be injective. Since we have assumed  $H^1(G; \mathbf{Z}G) \neq 0$  the result follows.

## 5. THE VANISHING OF $q(G)$

5.1. Here we mention in a few words what happens when for a finitely presented group  $G$  the invariant  $q(G)$  is 0. For the details and more comments we refer to the paper [E2]. We thus consider a 4-manifold  $M$  with  $\pi_1(M) = G$  and  $\chi(M) = 0$ .

Since we restrict attention to groups with  $\bar{\beta}_1(G) = 0$  the vanishing of  $\chi(M)$  implies  $\bar{\beta}_2(M) = 0$ , whence  $\bar{H}^2(\tilde{M}) = 0$ . As shown in [E2] by a spectral sequence argument it follows that  $H^2(M; \mathbf{Z}G)$  is isomorphic to  $H^2(G; \mathbf{Z}G)$ , ordinary cohomology with local coefficients  $\mathbf{Z}G$ . By Poincaré duality  $H^2(M; \mathbf{Z}G) = H_2(M; \mathbf{Z}G)$  which can be identified with  $H_2(\tilde{M}; \mathbf{Z})$ . Since  $\tilde{M}$  is simply connected,  $H_2(\tilde{M}; \mathbf{Z})$  is isomorphic to the second homotopy group  $\pi_2(\tilde{M}) = \pi_2(M)$ .

What about  $H_3(\tilde{M}; \mathbf{Z})$ ? It can be identified with  $H_3(M; \mathbf{Z}G)$  which, by Poincaré duality, is  $\cong H^1(M; \mathbf{Z}G) = H^1(G; \mathbf{Z}G)$ . This group, the “endpoint-group” of  $G$ , is known to be either 0 or  $\mathbf{Z}$  or of infinite rank. As mentioned in 4.4, remark 3) the latter case is excluded by our assumption  $\bar{\beta}_1(G) = 0$ . The case  $H^1(G; \mathbf{Z}G) = \mathbf{Z}$  is exceptional: it means that  $G$  is virtually infinite cyclic, and we exclude this. Then  $H_3(\tilde{M}; \mathbf{Z}) = 0$ .

5.2. We now add the assumption that  $H^2(G; \mathbf{Z}G) = 0$ . This is a property shared by many groups (e.g. duality groups). Then the homology groups  $H_i(\tilde{M}; \mathbf{Z})$  are  $= 0$  for  $i = 1, 2, 3, 4$  ( $i = 4$  because  $\tilde{M}$  is an open manifold). Thus all homotopy groups of  $\tilde{M}$  are  $= 0$ ,  $\tilde{M}$  is contractible,  $M$  is a  $K(G, 1)$ , and the group  $G$  fulfills Poincaré duality.

**THEOREM 6.** *Let  $G$  be an infinite, finitely presented group, not virtually infinite cyclic, fulfilling  $\bar{\beta}_1(G) = 0$  and  $H^2(G; \mathbf{Z}G) = 0$ , and let  $M$  be a manifold with fundamental group  $G$ . If the Euler characteristic  $\chi(M) = 0$ , then  $M$  is an Eilenberg-MacLane space for  $G$  and  $G$  is a Poincaré duality group of dimension 4.*

We recall that for knot groups and 2-knot groups  $q(G) = 0$ , see examples 3) and 4) in 2.2. Theorem 6 can only be applied to 2-knot groups which are not classical knot groups since the latter have cohomological dimension 2.