# 2. A Projective Threefold with a Nodal Cubic as Cup Form

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ii) *Let $C \subset Y$ be a smooth curve, and $\sigma\colon X \longrightarrow Y$ be the blow up of $Y$ along this curve. Using the same notation as in* i), *the cup form of $X$ is described by the polynomial*

$$q(x_1, \dots, x_n) - 3 \cdot \Big(\sum_{i=1}^{n} (C \,.\, \kappa_i) x_i x_0^2\Big) - \deg_C(N_{C/Y}) x_0^3 \,.$$

*Here, $C \,.\, \kappa_i$ stands for the evaluation of the homology class of $C$ on $\kappa_i$, and $N_{C/Y}$ is the normal bundle of $C$ in $Y$.*

*Proof.* This follows easily from [GH], p.602ff. $\square$

1.3. *Complete Intersections in Products of Projective Spaces.*

Let $\mathbf{P}_{n_1} \times \dots \times \mathbf{P}_{n_r}$ be a product of projective spaces. Write $\mathcal{O}(a_1, \dots, a_r)$ for the invertible sheaf $\pi_1^* \mathcal{O}_{\mathbf{P}_{n_1}}(a_1) \otimes \dots \otimes \pi_r^* \mathcal{O}_{\mathbf{P}_{n_r}}(a_r)$. Here, $\pi_i$ is the projection onto the $i$-th factor. If all the $a_i$'s are positive, this sheaf is very ample. A section in it is given by a multihomogeneous polynomial of multidegree $(a_1, \dots, a_r)$. We denote by

$$\left[\begin{array}{c|ccc} \mathbf{P}_{n_1} & a_1^1 & \dots & a_1^m \\ \vdots & \vdots & & \vdots \\ \mathbf{P}_{n_r} & a_r^1 & \dots & a_r^m \end{array}\right]$$

the family of zero sets of sections of the sheaf

$$\mathcal{O}(a_1^1, \dots, a_r^1) \oplus \dots \oplus \mathcal{O}(a_1^m, \dots, a_r^m) \,.$$

The members of this family are complete intersections of $m$ hypersurfaces. An iterated application of Theorem 5 shows that a general member $X$ of such a family is smooth and simply connected and that $(h_1, \dots, h_m)$ with $h_i := \pi_i^*(c_1(\mathcal{O}_{\mathbf{P}_{n_i}}(1)))$ is a basis for $H^2(X, \mathbf{Z})$.

## 2. A Projective Threefold with a Nodal Cubic as Cup Form

Let $Y$ be a smooth member of the family $\left[\begin{array}{c|cc} \mathbf{P}_4 & 1 & 2 \\ \mathbf{P}_1 & 1 & 1 \end{array}\right]$. We first compute the cup form of $Y$. Let $(\widetilde{h}_1, \widetilde{h}_2)$ be the canonical basis of $H^2(\mathbf{P}_4 \times \mathbf{P}_1, \mathbf{Z})$, and $(h_1, h_2)$ be the basis of $H^2(X, \mathbf{Z})$ as described in 1.3. We compute, e.g.,

$$h_1^2 h_2 = \widetilde{h}_1^2 \widetilde{h}_2 (\widetilde{h}_1 + \widetilde{h}_2)(2\widetilde{h}_1 + \widetilde{h}_2) = 2\widetilde{h}_1^4 \widetilde{h}_2 = 2 \,.$$

Here we have written the cup product followed by evaluation on the fundamental class as multiplication. The cup form of $Y$ is given by the polynomial

$$3x_1^3 + 6x_1^2 x_2 \,.$$

$Y$ contains four smooth curves $C_i \cong \mathbf{P}_1$, $i = 1, \ldots, 4$, such that $C_i \,.\, h_1 = 0$, $C_i \,.\, h_2 = 1$, and $N_{C_i/X} \cong \mathcal{O}_{\mathbf{P}_1}(-1) \oplus \mathcal{O}_{\mathbf{P}_1}(-1)$. To see this, observe that $Y$ is defined by two equations $l_0 \cdot x_0 + l_1 \cdot x_1 = 0$ and $q_0 \cdot x_0 + q_1 \cdot x_1 = 0$. Here, $x_0$ and $x_1$ are the homogeneous coordinates of $\mathbf{P}_1$ and $l_0, l_1$ and $q_0, q_1$ are linear and quadratic homogeneous polynomials in 5 variables (the homogeneous coordinates of $\mathbf{P}_4$). It is easily computed that the image of $Y$ under the projection to $\mathbf{P}_4$ is the hypersurface $\widetilde{Y} := \{l_0 q_1 - l_1 q_0 = 0\}$. For a generic choice of $l_0, l_1, q_0, q_1$, the set $S := Z(l_0, l_1, q_0, q_1)$ consists of 4 points (Thm. 5). It is obvious that the projection $Y \longrightarrow \widetilde{Y}$ is an isomorphism above $\widetilde{Y} \setminus S$ and that the fibre above a point in $S$ is of the type $\{s\} \times \mathbf{P}_1$. The description of the normal bundle is a consequence of this. Let $X$ be the blow up of $Y$ in one of these curves. By Theorem 6, the cup form of $X$ is given by the polynomial

$$3x_1^3 + 6x_1^2 x_2 - 3x_0^2 x_2 + 2x_0^3 .$$

This defines an irreducible plane cubic with a node.

## 3. Quaternary Cubic Forms that are Cup Forms of Projective Algebraic Manifolds

On the one hand, we know by [OV], Prop. 16, that cubic forms whose Hessian vanishes identically cannot occur as cup forms of projective threefolds. The Hessian of a quaternary cubic form $f$ vanishes identically if and only if the surface $f = 0$ is a cone over a plane cubic curve. On the other hand, we have collected a number of families in which we find cup forms of simply connected projective threefolds. There are some families which are not covered by these two results, for them the problem of realizability remains unsolved.

THEOREM 7. *There are polynomials occurring as cup forms of projective algebraic manifolds in the following families of non-singular forms:*

$$(*), \quad (*_1), \quad (*_2), \quad (*_4) \ \textit{and} \ (*_5),$$

*and in the following families of forms defining surfaces with isolated singularities:*

$$(A_1),\ (2A_1),\ (3A_1),\ (4A_1),\ (2A_1A_2),\ (A_2),\ (2A_2),\ (3A_2) \ \textit{and} \ (D_4^{II}).$$

*Furthermore, forms which define in $\mathbf{P}_3$ the union of a non-singular quadric with a transversal plane, or the union of a quadric cone with a transversal plane can be realized as cup forms of projective algebraic manifolds.*