5. A WORKED EXAMPLE

Important work of Gabai and Kazez [5], [6] which uses three-dimensional topology shows that, if $\alpha_1, \alpha_2 \colon G_1 \to G_2$ are nonzero-degree homomorphisms between infinite surface groups, they are strongly equivalent if and only if $\mathcal{G}(\alpha_1) = \mathcal{G}(\alpha_2)$, $\alpha_1(G_1) = \alpha_2(G_1)$ and $\alpha_1(G_1^+) = \alpha_2(G_1^+)$. They also show that, if $\alpha_1, \alpha_2 \colon G_1 \to G_2$ are homomorphisms between surface groups at least one of which is finite, then α_1, α_2 are strongly equivalent if and only if $\mathcal{G}(\alpha_1) = \mathcal{G}(\alpha_2)$, $\alpha_1(G_1) = \alpha_2(G_1)$ and $\alpha_1(G_1^+) = \alpha_2(G_1^+)$.

5. A WORKED EXAMPLE

In this section we will apply the algorithm to a rather trivial example to illustrate the algebraic manipulations involved.

Consider the homomorphism $\alpha \colon \langle a, b, c, d \mid (a,b)(c,d) \rangle \to \langle x, y \mid (x,y) \rangle$ induced by the homomorphism of free groups $A \colon \langle a, b, c, d \mid \ \rangle \to \langle x, y \mid \ \rangle$ determined by $(a,b,c,d) \mapsto (x, y, x, y^{-1})$.

We have

$$\begin{aligned} A((a,b)(c,d)) &= (x,y)(x,y^{-1}) = (x,y)x^{-1}yx(x,y)^{-1}x^{-1}y^{-1}x \\ &= (x,y)^{1-x^{-1}y^{-1}x}. \end{aligned}$$

Since α is orientation-true, Kneser's Theorem 4.8 implies that $\mathcal{G}(\alpha)$ is obtained by applying the orientation map to $1 - x^{-1}y^{-1}x$, so $\mathcal{G}(\alpha) = 0$. Thus we want to apply the algorithm to transform A into a map A' inducing α, such that $A'((a,b)(c,d)) = 1$.

Form the CW-surfaces associated with the given surface group presentations, so the free group generators can be viewed as loops.

Let us subdivide y into two edges, one again called y, and the other called z. We will call the vertices u and v, so that x is a loop at v, y joins v to u, and z joins u to v. The algorithm requires us to subdivide x, but, in order to keep the example simple, we shall not do this. Now we subdivide b and d into two edges labelled $y1$, $z1$, and $y2$, $z2$ respectively. Here the first letter indicates the image label, while, since we plan to depict the moves in planar diagrams, we also want a label to identify equal edges, and it is convenient to use integers for this identification. Similarly, we label c and d as $x1$ and $x2$, respectively.

We first use Construction 2.9 to get a cellular map, and hence a diagram, and then, after some simple applications of Construction 3.5 and 3.11, we can obtain the first diagram in Figure 5.1. Now we can apply the two-step

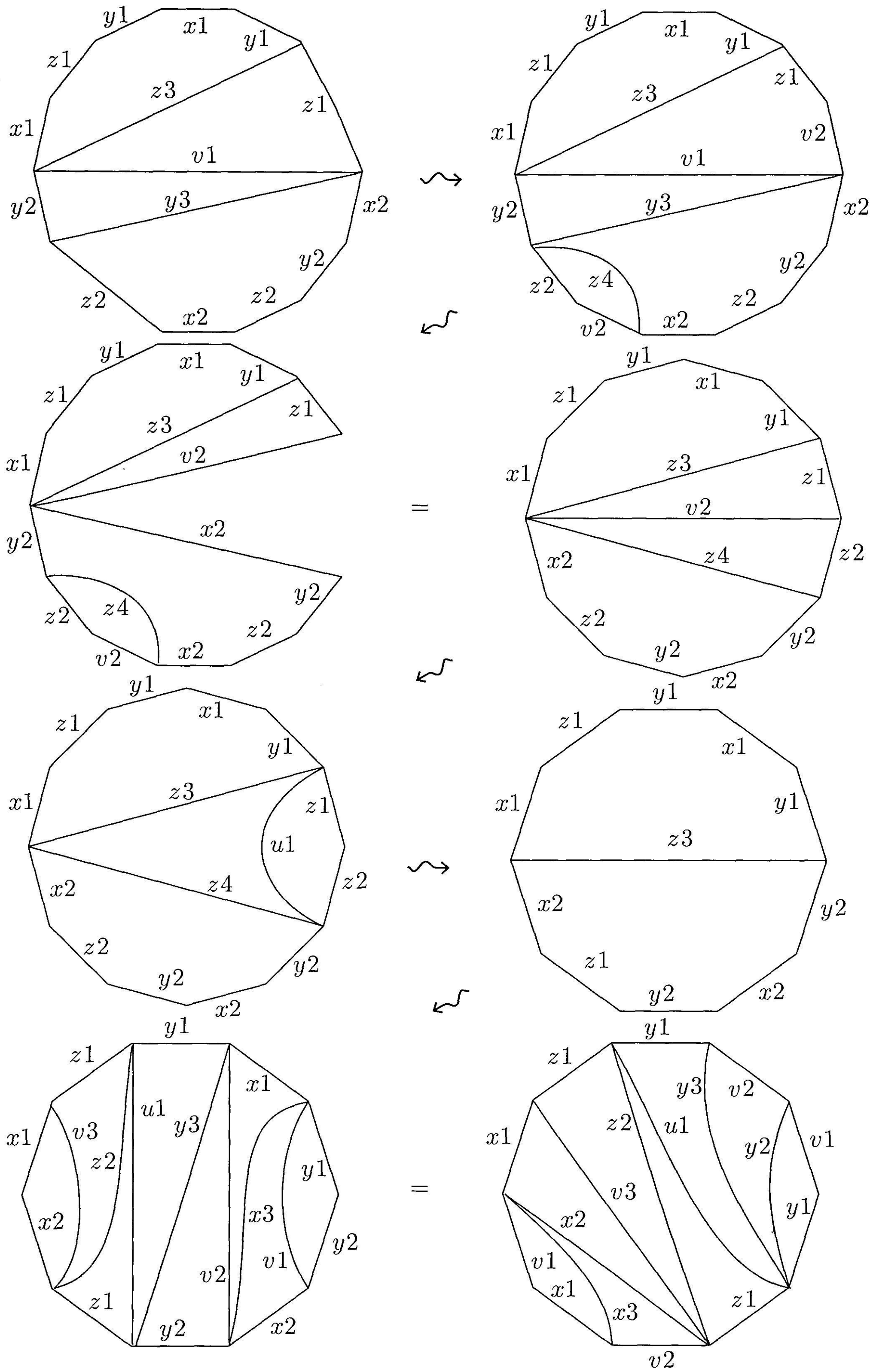

FIGURE 5.1
A worked example

Construction 3.18, to pass from the first to the second, and the second to the third diagram. Thus the first and third diagrams are obtained from the second, by first collapsing $v1$ and $v2$, respectively, and then identifying $z2 = z4$, and $y2 = y3$, respectively. The fourth diagram is a convenient redrawing of the third diagram. Now we apply Construction 3.16 (a) to identify $z1 = z2$ and $z3 = z4$, and arrive at the sixth diagram in Figure 5.1.

Now we can apply Construction 3.24 to arrive at the seventh diagram in Figure 5.1, where we have three punctured spheres, as depicted in Figure 5.2, and we see that $v1$, $v2$ are non-separating non-trivial V-loops, and cutting along these leaves a sphere with four punctures, which can be opened up into a disc by cutting along $x1$, $y1$, and $z1$. Thus we can rearrange the seventh diagram in Figure 5.1 to obtain the eighth diagram.

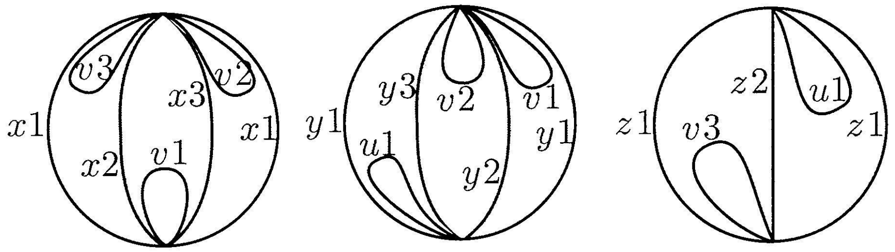

FIGURE 5.2
A normal form

To see what this says about our original group homomorphism, we express all the steps algebraically, by manipulating groupoid presentations.

Here X_2 has only one face, and we have to choose a maximal subtree, and it is natural to choose $\{u, v, z\}$. Let us express this by writing $\langle x, y; z \mid (x, yz)\rangle$, where the edges after the semicolon specify (the edge set of) a maximal subtree among the boundary edges. Recall that for CW-surface fundamental groupoid presentations we do not specify vertices, since they correspond to face-adjacency cycles.

In the same spirit, we express the first diagram in Figure 5.1 as

$$\langle x1, y1, x2, y2; z1, z2; v1, y3, z3 \\ \mid \overline{x1}\,\overline{z1}\,\overline{y1}x1y1z3,\ \overline{z3}\,z1v1, \overline{v1}y3\overline{y2},\ \overline{y3}\,\overline{x2}y2z2x2\overline{z2}\rangle,$$

where the edges after the second semicolon specify the edges to be erased to form a single face, and overlines indicate inverses. Here we can identify $a = x1$, $b = y1z1$, $c = x2$, $d = \overline{z2}\,\overline{y2}$.

Now we introduce a new vertex, two new edges $v2$, $z4$, and a new face $z4 = z2v2$, so the second diagram is expressed as

$$\langle x1, y1, x2, y2; z1, z2, v2; v1, y3, z3, z4 \\ \mid \overline{x1}\,\overline{z1}\,\overline{y1}x1y1z3,\ \overline{z3}\,z1v2v1, \overline{v1}y3\overline{y2},\ z4\overline{v2}\,\overline{z2},\ \overline{y3}\,\overline{x2}y2z2x2\overline{z4}\rangle,$$

and here $x2$ is an arc, and we identify $a = x1$, $b = y1z1$, $c = x2\overline{v2}$, $d = \overline{z2}\,\overline{y2}$. (We can recover the first diagram by collapsing $v2$ in the maximal subtree, and cashing in the new face relation to identify $z4 = z2$.)

We now collapse the edge $v1$ in the maximal subtree, and cash in the face relation $(v1)y2 = y3$, to identify the two eges $y2 = y3$. This obliges us to choose new edges to erase, and we find that the fourth, and third, diagrams are expressed as

$$\langle x1, y1, x2, y2; z1, z2; z3, z4, v2 \\ \mid \overline{x1}\,\overline{z1}\,\overline{y1}x1y1z3,\ \overline{z3}\,z1v2, z4\overline{v2}\,\overline{z2},\ \overline{y2}\,\overline{x2}y2z2x2\overline{z4}\rangle,$$

and here we identify $a = x1$, $b = y1z1$, $c = x2\overline{v2} = \overline{z2}\,\overline{y2}\,x2y2z2$, $d = \overline{z2}\,\overline{y2}$.

We now re-triangulate, and the fifth diagram can be expressed as

$$\langle x1, y1, x2, y2; z1, z2; z3, z4, u1 \\ \mid \overline{x1}\,\overline{z1}\,\overline{y1}x1y1z3,\ \overline{z3}\,\overline{u1}\,z4, u1z1\overline{z2},\ \overline{y2}\,\overline{x2}y2z2x2\overline{z4}\rangle,$$

and here we identify $a = x1$, $b = y1z1$, $c = \overline{z2}\,\overline{y2}\,x2y2z2$, $d = \overline{z2}\,\overline{y2}$.

We now collapse the edge $u1$, and make identifications using the face relations $z2 = (u1)z1$, $z4 = (u1)z3$, and the sixth diagram can be expressed as $\langle x1, y1, x2, y2; z1; z3 \mid \overline{x1}\,\overline{z1}\,\overline{y1}x1y1z3, \overline{y2}\,\overline{x2}y2z1x2\overline{z3}\rangle$, and here we identify $a = x1$, $b = y1z1$, $c = \overline{z1}\,\overline{y2}\,x2y2z1$, $d = \overline{z1}\,\overline{y2}$.

We now retriangulate, to express relations which map to relations in the free group.

Notice that we have now lifted α to the homomorphism

$$A' \colon \langle a, b, c, d \mid \ \rangle \to \langle x, y \mid \ \rangle$$

determined by $(a, b, c, d) \mapsto (x, y, \bar{y}xy, \bar{y})$, and $A'((a, b)(c, d)) = 1$.

Moreover, by changing presentations, we can now express α in a more natural form. We take the non-separating v-loops $v1 = y1\overline{y2}$ and $v2 = x1y1\overline{y2x2}$, and get the presentation

$$\langle x1, y1, v1, v2; z1 \mid \overline{x1}\,\overline{z1}, \overline{y1}v2\overline{v1}y1z1\overline{v2}x1v1\rangle,$$

and here we identify $x2 = \overline{v2}x1v1$, $y2 = \overline{v1}y1$, so $a = x1$, $b = y1z1$, $c = \overline{z1}\,\overline{y2}\,x2y2z1 = \overline{z1}\,\overline{y1}v1\overline{v2}x1y1z1$, $d = \overline{z1}\,\overline{y2} = \overline{z1}\,\overline{y1}v1$.

Now we can collapse the maximal subtrees, and we have a description of our group homomorphism as follows. We have the genus two surface group $\langle x1, y1, v1, v2 \mid \overline{x1}\,\overline{y1}v2\overline{v1}y1\overline{v2}x1v1\rangle$, we first impose relations annihilating the two generators $v1$, $v2$, to get a free group, and we then impose a relation to get the genus one surface group. Here we can identify $a = x1$, $b = y1$, $c = \overline{y1}v1\overline{v2}x1y1$, $d = \overline{y1}v1$, and thus $bd = v1$, $ab\bar{c}d = v2$. This represents α in one of the normal forms described in Case 4.2.

ACKNOWLEDGEMENTS. We thank Allan Edmonds, Rostislav Grigorchuk, John Guaschi, Boju Jiang, Gilbert Levitt and Joan Porti for information about the literature. We especially thank Carles Casacuberta for showing us Example 3.33, and the referee for much useful advice.

The first author is grateful to the DGICYT (Spain) for support through grant PB93-0900. The second author is grateful to the Centre de Recerca Matemàtica, Barcelona, for hospitality during the Spring term of 1994.

REFERENCES

[1] DICKS, W. and M. J. DUNWOODY. *Groups Acting on Graphs.* Cambridge Studies in Advanced Mathematics, vol. 19. Cambridge University Press, 1989.

[2] DICKS, W. and E. FORMANEK. Automorphism subgroups of finite index in algebraic mapping class groups. *J. Algebra 189* (1997), 58–89.

[3] EDMONDS, A. L. Deformation of maps to branched coverings in dimension two. *Ann. of Math. (2) 110* (1979), 113–125.

[4] GABAI, D. The simple loop conjecture. *J. Differential Geom. 21* (1985), 143–149.

[5] GABAI, D. and W. H. KAZEZ. The classification of maps of surfaces. *Invent. Math. 90* (1987), 219–242.

[6] GABAI, D. and W. H. KAZEZ. The classification of maps of nonorientable surfaces. *Math. Ann. 281* (1988), 687–702.

[7] GRIGORCHUK, R. I. and P. F. KURCHANOV. The classification of epimorphisms from fundamental groups of surfaces to free groups. *Mat. Zametki 48* (1990), 26–35.

[8] GRIGORCHUK, R. I., P. F. KURCHANOV and H. ZIESCHANG. Equivalence of homomorphisms of surface groups to free groups and some properties of 3-dimensional handlebodies. *Cont. Math. 131* (1992), 521–530.

[9] HANSEN, V. L.(ed.) *Jakob Nielsen: Collected Mathematical Papers, Volume 1 (1913-1932).* Birkhäuser, 1986.

[10] KNESER, H. Glättung von Flächenabbildungen. *Math. Ann. 100* (1928), 609–616.

[11] —— Die kleinste Bedeckungszahl innerhalb einer Klasse von Flächenabbildungen. *Math. Ann. 103* (1930), 347–358.

[12] MAUNDER, C. R. F. *Algebraic Topology.* Cambridge University Press, 1980.

[13] NEISENDORFER, J. Primary homotopy theory. *Mem. Amer. Math. Soc. 25* (1980).

[14] NIELSEN, J. Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen. *Acta Math. 50* (1927), 189–358.

[15] OL'SHANSKII, A. Y. Diagrams of homomorphisms of surface groups. *Siberian Math. J. 30* (1989), 961–979.

[16] SKORA, R. The degree of a map between surfaces. *Math. Ann. 276* (1987), 415–423.