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Autor(en): **Wingren, Peter**

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CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

by Peter WINGREN

ABSTRACT. A real-valued continuous nowhere-differentiable function on [0, 1] is constructed. Its graph F is proved to have the following property. If B is a Borel subset of F and if the projection of B on [0, 1] has positive Lebesgue measure, then the Hausdorff dimension of B is two.

0. INTRODUCTION

In 1903 Takagi [TAK, p. 176] gave an extremely simple construction of a nowhere differentiable real-valued continuous function on [0, 1]. Takagi's construction is

$$T(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^p x, \mathbf{Z}) \tag{1}$$

where each term is a scaled version of the sawtooth function

$$\operatorname{dist}(x, \mathbf{Z}) := \inf\{ |x - y| : y \in \mathbf{Z}\} . \tag{2}$$

Later, in 1930, van der Waerden [WAE] gave a similar example, which de Rham [RHA], in 1957, improved to an example identical with Takagi's.

It follows from a proof of Mauldin and Williams [M-W, pp. 795-797] that the graph of the Takagi function has a σ-finite linear Hausdorff measure and hence is of Hausdorff dimension 1.

In 1937 Besicovitch and Ursell [B-U, p. 29] constructed for an arbitrary α, $1 < \alpha < 2$, a real-valued nowhere-differentiable function in $C[0, 1]$ with graph of Hausdorff dimension α. They too used the sawtooth function $\operatorname{dist}(x, \mathbf{Z})$ as a building block in their construction.

In this paper we construct a real valued continuous function $f(x), x \in [0, 1]$, whose graph has an optimal property with respect to Hausdorff dimension and measure.

We prove that for an arbitrary α, $1 < \alpha < 2$, $f(x)$ has the property

$\mathscr{P}(\alpha)$: Every Borel subset $B \subset \text{graph}(f)$, with projection on the x-axis of positive Lebesgue measure $m(\text{Proj}(B)) > 0$, has infinite α-dimensional Hausdorff measure

$$H^{\alpha}(B) = +\infty . \tag{3}$$

It is easy to see that

$$\mathscr{P}(\alpha)\ \forall \alpha < 2 \Leftrightarrow \mathscr{P}$$

where

$\mathscr{P}$: Every Borel set $B \subset \text{graph}(f)$ with $m(\text{Proj}(B)) > 0$ has Hausdorff dimension equal to two.

Rather than establish a general theorem valid for a class of functions we shall construct a single function with the desired property. The rationale is to provide a simple construction accompanied by a short, clear and instructive proof.

Our function is

$$f(x) = \sum_{p=0}^{\infty} 2^{-p} \text{dist}(2^{2^p} x, \mathbf{Z}) . \tag{4}$$

Even though $\mathscr{P}$ is established for only a single function f, the proof contains general methods extracted as Lemma 1 and Lemma 2. It appears that Lemma 1 is well known in more general cases than ours; compare [P-U, p. 159, the beginning of the proof of their Lemma 1]. However the proof is included here for completeness and because in the present case it is particularly simple.

The author is grateful to Professor V.P. Havin [HAV] for suggesting the investigation of fractal graphs with respect to $\mathscr{P}(\alpha)$, $\alpha = 1$.

PROBLEM. We believe that the following problem is unsolved.

Part 1: Construct a real valued function in $C[0, 1]$ with graph of Hausdorff dimension 1 and with property $\mathscr{P}(\alpha)$ for $\alpha = 1$.

Part 2: Determine the optimal smoothness in terms of the second difference of such a function.

Notation. The diameter of U is denoted by $|U|$ and the L^1-norm of $g \in L^1(\mathbf{R})$ by $\|g\|$. If f is a real valued function in $C[0, 1]$, we write $\tilde{f}(x)$ for $(x, f(x))$. The notation $H^{\alpha}(F)$ stands for α-dimensional Hausdorff measure of a set $F \subset \mathbf{R}^2$ and $M^{\alpha}(F)$ is the α-dimensional net measure of F

constructed by closed dyadic cubes. The graph of a real valued function $f \in C[0, 1]$ is denoted by graph (f). By a dyadic cube we mean a cube which is the Cartesian product of dyadic intervals. If Q is an arbitrary dyadic closed cube, then the band of type $\{(x, y) : (x, z) \in Q$ for some $z \in \mathbf{R}\}$ is called a dyadic band. In our construction the dyadic bands of width 2^{-2^p} play a special role. They are called bands of generation $p, p = 0, 1, 2, \ldots$.

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1. A LEMMA ABOUT MASS DISTRIBUTION

By a mass distribution on a subset A of $\mathbf{R}^2$ we mean a measure μ on A such that $0 < \mu(A) < \infty$.

LEMMA 1. *Let f be a real valued measurable function defined on $[0, 1]$. Then there is a mass distribution μ on $F := \mathrm{graph}(f)$ such that*

1) *for any two subintervals I and I' of $[0, 1]$, with $m(I) = m(I')$,*

$$\mu(I \times \mathbf{R}) = \mu(I' \times \mathbf{R})$$

and

2) *if for two Borel sets B_1 and B_2 in $[0, 1] \times \mathbf{R}$ there exists $(x_0, y_0) \in \mathbf{R}^2$ such that*

$$B_1 \cap F + (x_0, y_0) = B_2 \cap F$$

then

$$\mu(B_1) = \mu(B_2) .$$

Proof. Let B be an arbitrary Borel set in $\mathbf{R}^2$. Define

$$\mu(B) = m(\tilde{f}^{-1}(B)) . \tag{5}$$

Then it is obvious that μ is a mass distribution on graph(f) and 1) and 2) follow from the translation invariance of the Lebesgue measure.

2. A LEMMA ABOUT MASS DISTRIBUTION AND SUCCESSIVE TRANSLATIONS

LEMMA 2. *Let $g(y) \geqslant 0$ and $g(y) \in L^1(\mathbf{R})$. If I is a finite interval and d is a positive real number then*

$$\int_I \sum_{n=-\infty}^{\infty} g(y - nd)\,dy < \left(1 + \operatorname{int} \frac{m(I)}{d}\right) \cdot \| g \| . \tag{6}$$

Proof. It suffices to assume that $I = [0, m(I)]$. The general case will then follow by a change of variables. If we use the notation $M = \operatorname{int}\frac{m(I)}{d}$ we get

$$\int_I \sum_{n=-\infty}^{+\infty} g(y-nd)\,dy \leqslant \sum_{m=0}^{M} \int_{m\cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y-nd)\,dy$$

$$(7) \qquad = \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{m\cdot d}^{(m+1)d} g(y-nd)\,dy = \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t)\,dt$$

$$= \sum_{m=0}^{M} \| g \| = \left(1 + \operatorname{int}\left(\frac{m(I)}{d}\right)\right) \| g \| .$$

3. Hausdorff measure, net measure and Hausdorff dimension

This section presents standard results and definitions; see for example [FAL1].

The α-dimensional Hausdorff measure of a subset A of $\mathbf{R}^n$ is defined by

$$(8) \qquad H^{\alpha}(A) = \lim_{\delta \to 0} \inf_{\{U_i\}} \sum_{i=1}^{\infty} | U_i |^{\alpha} ,$$

where $\{U_i\}_1^{\infty}$ is a covering of A with $| U_i | < \delta$, $i = 1, 2, \ldots$, and the infimum is taken over all such coverings. The unique number α_0 such that $\alpha < \alpha_0$ implies $H^{\alpha}(A) = +\infty$ and $\alpha_0 < \alpha$ implies $H^{\alpha}(A) = 0$ is by definition the Hausdorff dimension of A.

The net measure $M^{\alpha}(A)$ of A is defined similarly except that the coverings $\{U_i\}$ consist of closed dyadic cubes. It follows that there exists a constant $c_1 > 0$ such that

$$(9) \qquad c_1 M^{\alpha}(A) \leqslant H^{\alpha}(A) \leqslant M^{\alpha}(A) .$$

Since $M^{\alpha}(A)$ and $H^{\alpha}(A)$ must therefore yield identical dimensions for A it will suffice to work with dyadic cubes.

4. Mass distribution and Hausdorff dimension

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.

Mass Distribution Principle. Let μ *be a mass distribution on* $A \subset \mathbf{R}^n$. *If there exist constants* $c > 0$ *and* $\delta > 0$ *such that, for all dyadic cubes* $Q \subset \mathbf{R}^n$ *with* $|Q| \leqslant \delta$,

$$\mu(Q) \leqslant c \cdot |Q|^{\alpha}, \tag{10}$$

then

$$\alpha \leqslant \dim_H(A). \tag{11}$$

Proof. Let $\{Q_i\}_{i=1}^{\infty}$ be a covering of A with dyadic cubes of diameter not exceeding δ. Then

$$0 < \mu(A) \leqslant \mu\left(\bigcup_{i=1}^{\infty} Q_i\right) \leqslant \sum_{1}^{\infty} \mu(Q_i) \leqslant c \cdot \sum_{1}^{\infty} |Q_i|^{\alpha} \tag{12}$$

and hence the discontinuity in the $M^{\alpha}(A)$-graph from $+\infty$ to 0 occurs at a value not less than α. Thus

$$\alpha \leqslant \dim_H(A). \tag{13}$$

5. The main result

The notation used in the following theorem and in its proof can be found in Section 0.

THEOREM. *Let*

$$f(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^{2^p} x, \mathbf{Z}), \quad x \in [0, 1]. \tag{14}$$

Then for every Borel subset B *of* graph(f) *with* $m(\operatorname{Proj}(B)) > 0$,

$$\dim_H(B) = 2. \tag{15}$$

Proof. Assume that B is a Borel set as above. From graph$(f) \subset \mathbf{R}^2$ there follows

$$\dim_H(B) \leqslant 2. \tag{16}$$

It will suffice to prove that

$$\alpha \leqslant \dim_H(B) \tag{17}$$

for an arbitrary positive $\alpha < 2$. Distribute the unit mass as in Lemma 1. Let Q be a dyadic cube with side length less than $\frac{1}{4}$. Then the side length of Q is 2^{-n} for some positive integer n and there is a positive integer p such that

$$2^{-2p+1} \leqslant 2^{-n} < 2^{-2p} . \tag{18}$$

Let D_0 be the smallest vertical band which inscribes Q, and so it has band width 2^{-n}. From the second inequality in (18) we conclude that D_0 is contained in a band from generation p. In the discussion and in the estimations which follow, just those bands which are of generations $p, p+1$ and $p+2$ play a role. We let D and D_L denote an arbitrary band from generation $p+1$ and its left half, respectively. On D_L we study $f(x)$ as a sum of two terms,

$$f(x) = \sum_{0}^{p+1} 2^{-k} \operatorname{dist}(2^{2^k}x, \mathbf{Z}) + \sum_{p+2}^{\infty} 2^{-k} \operatorname{dist}(2^{2^k}x, \mathbf{Z}) . \tag{19}$$

The first term is linear and the second periodic (one cycle on each subband from generation $p+2$). This implies that the distribution of mass via (5) on each $(p+2)$-subband (of D_L) is the same but translated a fixed distance d_D in y-direction. Now let D' be a $(p+2)$-subband of D_L and define

$$G_{D'}(y) := \mu(\{(x_1, x_2) \in D' \text{ and } x_2 \leqslant y\}) . \tag{20}$$

Then its derivative $g(y)$ exists a.e. and

$$\| g \| = 2^{-2p+2} . \tag{21}$$

If D' and D'' are neighbouring $(p+2)$-generation subbands of D_L, then $G_{D''}(y)$ is a translation of $G_{D'}(y)$ by d_D. Hence, we may use just one function G and its translates. The fixed translation d_D of mass in y-direction from one band to the next may be estimated by the derivative of the first sum of (19),

$$\begin{aligned} d_D &= 2^{-2p+2} \times \left| \frac{d}{dx} \left(\sum_{p=0}^{p+1} 2^{-k} \operatorname{dist}(2^{2^k}x, \mathbf{Z}) \right) \right| \\ &\geqslant 2^{-2p+2}(2^{-(p+1)+2^{p+1}} - 2^{-p+2^p} - \ldots - 2) > 2^{-2p+1-(p+2)} . \end{aligned} \tag{22}$$

The last inequality holds for $p > 1$, because the rapid decrease of the successive terms in the parenthesis implies that its value is larger than half the first term. (It is easy to check that this estimation also works for $(p+2)$-generation bands in the right band half D_R of a $(p+1)$-generation band).

Now consider the restrictions of $\tilde{f}(x)$ to all $(p+2)$-generation bands in D_L and D_R, and use the translation properties for G and its derivative g. Then by applying Lemma 2 with $\| g \| = 2^{-2p+2}$, $d > 2^{-2p+1-(p+2)}$, $m(I) = 2^{-n}$ we obtain

$$(23) \quad \mu(D_L \cap Q) + \mu(D_R \cap Q) \leqslant \left(1 + \text{int} \frac{2^{-n}}{2^{-2p+1-(p+2)}}\right) \cdot 2^{-2p+2}.$$

The number of bands from the $(p+1)$ generation contained in D_0 are $2^{-n}/2^{-2p+1}$, and, since $2^p < n$ by (18), we have, for $\alpha < 2$,

$$\mu(Q) = \mu(B_0 \cap Q) \leqslant \frac{2^{-n}}{2^{-2p+1}} \cdot \left(1 + \text{int} \frac{2^{-n}}{2^{-2p+1-(p+2)}}\right) \cdot 2^{-2p+2}$$

$$(24) \quad \leqslant 2^{-n} \cdot 2^{-2p+1} + 2^{-2n+p+2} \leqslant (2^{-n})^2 \cdot (1 + 2^{p+2})$$

$$\leqslant (2^{-n})^2 (1 + 4n) \leqslant (2^{-n})^\alpha = |Q|^\alpha$$

if $1 + 4n \leqslant 2^{n(2-\alpha)}$.

The Mass Distribution Principle now gives (17) and the proof is complete.

Remark. The nowhere-differentiability of the constructed function f is omitted in the statement of the Theorem. However this property can be established by minor changes to the proof in [RHA] or the proof of Theorem 2-9 in [D-W]. The continuity of $f(x)$ follows from uniform convergence of the series (4).

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Peter Wingren

Umeå University
Department of Mathematics
S-901 87 Umeå (Sweden)