### 4.3 EX AMPLES OF 1-CONNECTED NONKÄHLERIAN 3-FOLDS

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where $\xi=c_{1}\left(\mathscr{O}_{\mathbf{P}(E)}(1)\right), h \in H^{2}(Y, \mathbf{Z}), \quad$ and $\quad x \in \mathbf{Z}$. The other topological invariants of $\mathbf{P}(E)$ are:

$$
\begin{gathered}
\left.w_{2}(\mathbf{P}(E)) \equiv \pi^{*}\left(w_{2}(Y)+c_{1}(E)\right), p_{1}(E)\right) \\
=\pi^{*}\left[c_{1}(Y)^{2}-2 c_{2}(Y)+c_{1}(E)^{2}-4 c_{2}(E)\right], b_{3}(\mathbf{P}(E))=0 .
\end{gathered}
$$

Proof. The Leray-Hirsch theorem identifies the cohomology ring $H^{*}(\mathbf{P}(E), \mathbf{Z})$ with the ring $\left.H^{*}(Y, \mathbf{Z})[\xi] /<\xi^{2}+c_{1}(E) \cdot \xi+c_{2}(E)\right\rangle$; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence $0 \rightarrow \mathscr{O}_{\mathbf{P}(E)} \rightarrow \pi^{*} E \otimes \mathscr{O}_{\mathbf{P}(E)}(1) \rightarrow T_{\mathbf{P}(E)} \rightarrow \pi^{*} T_{Y} \rightarrow 0 . b_{3}(\mathbf{P}(E))=0$ follows from $b_{1}(Y)=0$ and the Leray-Hirsch theorem.

### 4.3 EXAMPLES OF 1-CONNECTED NON-KÄHLERIAN 3-FOLDS

Recall that the Hessian of a symmetric trilinear form $F \in S^{3} H^{\vee}$ on a free $\mathbf{Z}$-module $H$ of finite rank was defined as the composition $H_{F}: H \xrightarrow{F t} S^{2} H^{\vee} \xrightarrow{\text { disc }} \mathbf{Z}$. In terms of coordinates $\xi_{1}, \ldots, \xi_{b}$ on $H$ it is given by the determinant $\operatorname{det}\left(\frac{\partial^{2} f}{\partial \xi_{i} \xi_{j}}\right)$, where $f \in \mathbf{C}\left[H_{\mathbf{C}}\right]_{3}$ is the homogeneous cubic polynomial associated with $F$.

Proposition 16. Let $F$ be a symmetric trilinear form whose Hessian vanishes identically. Then $F$ is not realizable as cup-form of a Kählerian 3-fold.

Proof. Let $X$ be a complex 3-fold with a Kähler metric $g$. The Kähler class $\left[\omega_{g}\right] \in H^{2}(X, \mathbf{R})$ defines a multiplication map $\cdot\left[\omega_{g}\right]: H^{2}(X, \mathbf{R})$ $\rightarrow H^{4}(X, \mathbf{R})$, which is an isomorphism by the Hard Lefschetz Theorem $[\mathrm{G} / \mathrm{H}]$. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

Corollary 6. Cubic forms $f \in \mathbf{C}\left[H_{\mathbf{C}}\right]_{3}$ which depend on strictly less than $b=r k_{\mathbf{Z}} H$ variables are not realizable as cup-forms of Kählerian 3-folds with $b_{2}=b$.

By considering the Hessian of a cup-form over the reals one obtains further conditions.

DEFINITION 4. Let $F \in S^{3} H^{\vee}$ be a symmetric trilinear form on a free Z-module of rank $b$.

The Hesse cone of $F$ is the subset $\mathscr{H}_{F} \subset H_{\mathbf{R}}$ defined by $\mathscr{H}_{F}:=\left\{h \in H_{\mathbf{R}} \mid(-1)^{b} \operatorname{det}\left(F^{t}(h)\right)<0\right\}$.

The index cone $\mathscr{J}_{F}$ of $F$ is the subset $\mathscr{L}_{F}:=\left\{h \in \mathscr{H}_{F} \mid F^{t}(h) \in S^{2} H_{\mathbf{R}}^{\vee}\right.$ has signature $(1,-1, \ldots,-1)\}$.

Clearly $\mathscr{L}_{F}$ is an open subcone of $\mathscr{H}_{F}$ which coincides with $\mathscr{H}_{F}$ iff $b \leqslant 2$.

THEOREM 5. Let $F_{X} \in S^{3} H^{2}(X, \mathbf{Z})^{\vee}$ be the cup-form of a smooth projective 3-fold with $h^{0,2}(X)=0$. Then $F_{X}$ has a non-empty index cone.

Proof. Let $h \in H^{2}(X, \mathbf{Z})$ be the dual class of a hyperplane section $Y$ in some projective embedding. The inclusion $i: Y \hookrightarrow X$ induces a monomorphism $i^{*}: H^{2}(X, \mathbf{Z}) \rightarrow H^{2}(Y, \mathbf{Z})$ by the weak Lefschetz theorem. The symmetric bilinear form $F_{X}^{t}(h) \in S^{2} H^{2}(X, \mathbf{Z})^{\vee}$ is simply the pull-back of the cup-form of $Y$ under the inclusion $i^{*}$; it is therefore non-degenerate by the Hard Lefschetz theorem [L]. Applying the Hodge index theorem to $Y$ we see that the real bilinear form $F_{X}^{t}(h) \in S^{2} H^{2}(X, \mathbf{R})^{\vee}$ must have one positive and $b-1$ negative eigenvalues. In other words: $h \in I_{F_{X}}$.

REMARK 13. This result has two applications: it provides topological 'upper bounds' for the ample cone of a projective 3 -fold with $h^{0,2}=0$, and if gives further restrictions on symmetric trilinear forms to be realizable as cup-forms of projective 3 -folds with $h^{0,2}=0$ if $b \geqslant 4$.

These applications will be discussed in section 5 .
We will now describe examples of 1-connected, non-Kählerian, complex 3 -folds and determine their topological structure.

Example 10 (Calabi-Eckmann). E. Calabi and B. Eckmann have defined complex structures $X_{\tau}$, depending on a parameter $\tau$, on the product $S^{3} \times S^{3}[C / E]$. Their manifolds are principal fiber bundles over $\mathbf{P}^{1} \times \mathbf{P}^{1}$ whose fiber and structure group is the elliptic curve $E_{\tau}=\mathbf{C} /_{\mathbf{Z} \oplus \mathbf{Z}_{\tau}}, \operatorname{Im}(\tau)>0$.

The Calabi-Eckmann manifolds are homogeneous, non-Kählerian 3-folds of algebraic dimension 2.

Example 11 (Maeda). H. Maeda has generalized the Calabi-Eckmann construction. He constructed fiber bundles $X_{\tau}^{\prime}$ over Hirzebruch surfaces $\mathbf{F}_{n}, n \geqslant 0$, whose fiber and structure group are an elliptic curve $E_{\tau}$ and $\operatorname{Aut}\left(E_{\tau}\right)$ respectively $[\mathrm{M}] . X_{\tau}^{\prime}$ is again diffeomorphic to $S^{3} \times S^{3}$, and therefore non-Kählerian. Maeda's manifolds $X_{\tau}^{\prime}$ are homogeneous if and only if $n=0$ in which case they are Calabi-Eckmann 3-folds.

The Calabi-Eckmann construction can also be generalized in the following way:

Let $S^{2} \tilde{\times} S^{4}$ be the non-trivial $S^{4}$-bundle over $S^{2}$, i.e. $S^{2} \tilde{\times} S^{4}$ is the unique 1 -connected, closed, oriented, differentiable 6 -manifold with $H_{2}\left(S^{2} \widetilde{\times} S^{4}, \mathbf{Z}\right) \cong \mathbf{Z}$ and $b_{3}=0$, whose cup-form and Pontrjagin class vanish, but whose Stiefel-Whitney class $w_{2}$ is non-zero.

ThEOREM 6. For any integer $b \geqslant 0$ there exist compact complex 3-folds $X_{b}$, and $X_{b}^{\sim}$ if $b \geqslant 1$, which are homeomorphic to $\#_{b} S^{2} \times S^{4} \#_{b+1} S^{3} \times S^{3}$, and $S^{2} \overline{\times} S^{4} \#_{b-1} S^{2} \times S^{4} \#_{b+1} S^{3} \times S^{3}$.

Proof. Let $Y$ be a 1-connected, compact complex surface with $p_{g}(Y)=0$ and $b_{2}(Y) \geqslant 2$, and let $E=\mathbf{C} / \Gamma$ be the elliptic curve associated to the lattice $\Gamma \subset \mathbf{C}$. We want to construct the required 3-folds as total spaces of principal $E$-bundles over $Y$. Let $c: H_{2}(Y, \mathbf{Z}) \rightarrow \Gamma$ be an arbitrary epimorphism. The corresponding cohomology class $c \in H^{2}(Y, \Gamma)$ defines a topological principal bundle over $Y$ with fiber and structure group $E=\mathbf{C} / \Gamma$ as follows immediately from the identification of the classifying space $B E \simeq K(\Gamma, 2)$.

Let $\mathscr{O}_{Y}(E)$ be the sheaf of germs of holomorphic maps from $Y$ to $E$. We have a short exact sequence $0 \rightarrow \Gamma \rightarrow \mathscr{O}_{Y} \rightarrow \mathscr{O}_{Y}(E) \rightarrow 0$ and a corresponding exact cohomology sequence

$$
\rightarrow H^{1}\left(Y, \mathscr{O}_{Y}\right) \rightarrow H^{1}\left(Y, \mathscr{O}_{Y}(E)\right) \xrightarrow{\delta} H^{2}(Y, \Gamma) \rightarrow H^{2}\left(Y, \mathscr{O}_{Y}\right) \rightarrow
$$

By our assumptions $\delta$ is an isomorphism, so that every topological principal $E$-bundle admits a holomorphic structure. Let $X$ be the total space of such a bundle corresponding to a surjective map $c: H_{2}(Y, \mathbf{Z}) \rightarrow \Gamma$. The homotopy sequence of the fibration $p: X \rightarrow Y$ yields the sequence

$$
0 \rightarrow \pi_{2}(X) \xrightarrow{p_{*}} \pi_{2}(Y) \rightarrow \pi_{1}(E) \rightarrow \pi_{1}(X) \xrightarrow{p_{*}} \pi_{1}(Y) \rightarrow 0 .
$$

Since $Y$ is 1-connected, $\pi_{2}(Y)$ can be identified with $H_{2}(Y, \mathbf{Z})$, and then the boundary map $\pi_{2}(Y) \rightarrow \pi_{1}(E)$ becomes the characteristic map $c: H_{2}(Y, \mathbf{Z}) \rightarrow \Gamma$ of the bundle. This implies $\pi_{1}(X)=\{1\}$, whereas $H_{2}(X, \mathbf{Z})$ is given by: $0 \rightarrow H_{2}(X, \mathbf{Z}) \xrightarrow{p_{*}} H_{2}(Y, \mathbf{Z}) \xrightarrow{s} \Gamma \rightarrow 0$.

In particular, $H_{2}(X, \mathbf{Z})$ is free as a submodule of $H_{2}(Y, \mathbf{Z})$, and by dualizing the last sequence we obtain an identification (via $p^{*}$ )

$$
H^{2}(X, \mathbf{Z})=H^{2}(Y, \mathbf{Z}) /_{\Gamma^{v}} .
$$

The cup-form $F_{X}$ of $X$ is therefore trivial. In order to calculate $p_{1}(X)$ and $w_{2}(X)$, we use the exact sequence of tangent sheaves: $0 \rightarrow T_{X / X} \rightarrow T_{X}$
$\rightarrow p^{*} T_{Y} \rightarrow 0$. Since $T_{X / Y}$ is a trivial bundle, the characteristic classes of $X$ are simply the pullbacks of the corresponding classes of $Y$. But the $\operatorname{map} p^{*}: H^{4}(Y, \mathbf{Z}) \rightarrow H^{4}(X, \mathbf{Z})$ is zero, since $\left\langle p^{*}(\varepsilon) \cup p^{*}(\alpha),[X]\right\rangle$ $=<\varepsilon \cup \alpha, p_{*}[X]>=0$ for all classes $\varepsilon \in H^{4}(Y, \mathbf{Z})$, and $\alpha \in H^{2}(Y, \mathbf{Z})$.

Thus $p_{1}(X)=0$, and $w_{2}(X)$ is the residue of $w_{2}(Y) \in H^{2}\left(Y, \mathbf{Z}_{/ 2}\right)$ modulo $\Gamma^{\vee} / 2 \Gamma^{v}$.

The Euler characteristic of $X$ is zero, so that from $b_{2}(X)=b_{2}(Y)-2$ we find $b_{3}(X)=2\left(b_{2}(Y)-1\right)$. The system of invariants associated to the manifold $X$ is therefore given by

$$
\left(b_{2}(Y)-1, H^{2}(Y, \mathbf{Z}) /_{\Gamma^{v}}, w_{2}(Y)\left(\bmod \Gamma^{\vee} / 2 \Gamma^{v}\right), 0,0,0\right),
$$

i.e. $X$ is diffeomorphic to

$$
\# b_{2}(Y)-2 S^{2} \times S^{4} \# b_{2}(Y)-1 S^{3} \times S^{3} \text { if } w_{2}(Y) \in \Gamma^{\vee} / 2 \Gamma^{\vee},
$$

and to $S^{2} \tilde{\times} S^{4} \#_{b_{2}(Y)-3} S^{2} \times S^{4} \#_{b_{2}(Y)-1} S^{3} \times S^{3} \quad$ if $\quad b_{2}(Y) \geqslant 3$, and $w_{2}(Y) \notin \Gamma v / 2 \Gamma^{v}$.

Example 12 (Kato). In the two papers [K1], [K2] M. Kato studies the class of compact, complex 3 -folds $X$ containing smooth rational curves with neighborhoods biholomorphic to those of projective lines in $\mathbf{P}^{3}$. On this class of 3 -folds, called class $L$, he defines a semi-group structure + with neutral element $\mathbf{P}^{3}$.

Kato's connecting operation + is defined by removing 'lines' $L_{i} \subset X_{i}$ from 3 -folds $X_{i}, i=1,2$, and by identifying the complements $X_{i} \backslash L_{i}$ along open sets $U_{i} \backslash L_{i}$ obtained from suitable neighborhoods $U_{i} \subset X_{i}$.

Starting with a certain elliptic fiber space $X_{1}$ over the blow-up of $\mathbf{P}^{1} \times \mathbf{P}^{1}$ in a point, he constructs a sequence of 3 -folds $X_{n}:=X_{1}$ $+X_{n-1}, n \geqslant 2$. The 3 -folds $X_{n}$ are 1 -connected spin-manifolds with $H_{2}\left(X_{n}, \mathbf{Z}\right)=\mathbf{Z}$. Their cup-forms $F_{X_{n}}$, and their Pontrjagin classes $p_{1}\left(X_{n}\right)$ are in terms of a (normalized) generator $e_{n} \in H^{2}\left(X_{n}, \mathbf{Z}\right)$ and its dual class $\varepsilon_{n} \in H^{4}\left(X_{n}, \mathbf{Z}\right)$ given by $F_{X_{n}}\left(x e_{n}\right)=(n-1) x^{3}$, and $p_{1}\left(X_{n}\right)$ $=4(n-1) \varepsilon_{n}\left(\varepsilon_{n}\left(e_{n}\right)=1\right)$. The third Betti-number of $X_{n}$ is $4 n$.

In particular, $X_{1}$ is diffeomorphic to $S^{2} \times S^{4} \#_{2} S^{3} \times S^{3}$, and $X_{2}$ is diffeomorphic to $\mathbf{P}^{3} \#_{4} S^{3} \times S^{3}$. It is interesting to note that the Chernnumbers $c_{1}^{3}, c_{1} c_{2}$ of the $X_{n}$ are $c_{1}^{3}=64(1-n), c_{1} c_{2}=24(1-n)$, i.e. they satisfy $8 c_{1} c_{2}=3 c_{1}^{3}$. For projective manifolds of general type this equality is characteristic for ball quotients [Y].

EXAMPLE 13 (Twistor spaces). Let $p: Z \rightarrow M$ be the twistor fibration of a closed, oriented Riemannian 4-manifold ( $M, g$ ). $Z$ carries a natural almost complex structure which is integrable if and only if $g$ is self-dual $[\mathrm{A} / \mathrm{H} / \mathrm{S}]$.

Examples of 1-connected 4-manifolds which admit self-dual structures are $S^{4}, \#_{n} \mathbf{P}^{2}$, and $K 3$-surfaces.

The total spaces of their twistor fibrations are 1-connected complex 3-folds which may be Moishezon for $S^{4}$ and $\#_{n} \mathbf{P}^{2}$ [C], but which are usually non-Kähler [Hi]. We leave it to the reader to calculate the topological invariants of these 3 -folds. There is an interesting relation between Twistor spaces of connected sums and Kato's connection operation + for class $L$ manifolds [K2], [D/F].

EXAmple 14 (Oguiso). In a recent preprint [O1] K. Oguiso constructs examples of 1 -connected, Moishezon Calabi-Yau 3 -folds with very interesting cup-forms. He proves that for every integer $d \geqslant 1$ there exists a smooth complete intersection $X_{d}^{\prime}$ of type $(2,4)$ in $\mathbf{P}^{5}$ which contains a non-singular rational curve $C_{d}$ of degree $d$ with normal bundle $N_{C_{d} / X_{d}}=\mathscr{O}_{C_{d}}(-1)^{\oplus 2}$.

The 3 -fold $X_{d}^{\prime}$ can now be flopped along $C_{d}$, i.e. $C_{d}$ can be blown up to $\mathbf{P}\left(N_{C_{d} / X_{d}}\right) \cong \mathbf{P}^{1} \times \mathbf{P}^{1}$, and then 'blown down in the other direction'. The resulting 3 -fold $X_{d}$ is a 1-connected Moishezon manifold with trivial canonical bundle and cup-form $F_{X_{d}}$ given by $F_{X_{d}}\left(x e_{d}\right)=\left(d^{3}-8\right) x^{3}$. Here $e_{d} \in H^{2}\left(X_{d}, \mathbf{Z}\right)$ is the normalized generator corresponding to the strict transform of the negative of a hyperplane section of $X_{d}^{\prime}$. The Pontrjagin class of $X_{d}$ is $p_{1}\left(X_{d}\right)=(112+4 d) \varepsilon_{d}$ where $\varepsilon_{d} \in H^{4}\left(X_{d}, \mathbf{Z}\right)$ denotes the generator with $\varepsilon_{d}\left(e_{d}\right)=1$. Since the Euler-number does not change under a flop we have $b_{3}\left(X_{d}\right)=180$ for every $d$.

## 5. COMPLEX 3-FOLDS WITH SMALL $b_{2}$

In this section we investigate the following natural problem: Which cubic forms can be realized as cup-forms of compact complex 3-folds? For small $b_{2}$ something can be said: Any core of a 1-connected, closed, oriented differentiable 6-manifold with $H_{2}(X, \mathbf{Z}) \cong \mathbf{Z}$ is homotopy equivalent to the core of a 1-connected complex 3 -fold. In the case $b_{2}=2$, at least every discriminant $\Delta$ is realizable by a complex manifold. If $b_{2}=3$ we can realize all types of complex cubics with one exception, the union of a smooth conic and a tangent line. In addition to these realization results we prove a finiteness

