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where  $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ ,  $h \in H^2(Y, \mathbf{Z})$ , and  $x \in \mathbf{Z}$ . The other topological invariants of  $\mathbf{P}(E)$  are:

$$\begin{aligned} w_2(\mathbf{P}(E)) &\equiv \pi^*(w_2(Y) + c_1(E)), p_1(E)) \\ &= \pi^*[c_1(Y)^2 - 2c_2(Y) + c_1(E)^2 - 4c_2(E)], b_3(\mathbf{P}(E)) = 0. \end{aligned}$$

*Proof.* The Leray-Hirsch theorem identifies the cohomology ring  $H^*(\mathbf{P}(E), \mathbf{Z})$  with the ring  $H^*(Y, \mathbf{Z})[\xi]/\langle \xi^2 + c_1(E) \cdot \xi + c_2(E) \rangle$ ; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence  $0 \rightarrow \mathcal{O}_{\mathbf{P}(E)} \rightarrow \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \rightarrow T_{\mathbf{P}(E)} \rightarrow \pi^*T_Y \rightarrow 0$ .  $b_3(\mathbf{P}(E)) = 0$  follows from  $b_1(Y) = 0$  and the Leray-Hirsch theorem.

#### 4.3 EXAMPLES OF 1-CONNECTED NON-KÄHLERIAN 3-FOLDS

Recall that the Hessian of a symmetric trilinear form  $F \in S^3 H^\vee$  on a free  $\mathbf{Z}$ -module  $H$  of finite rank was defined as the composition  $H_F: H \xrightarrow{F^t} S^2 H^\vee \xrightarrow{\text{disc}} \mathbf{Z}$ . In terms of coordinates  $\xi_1, \dots, \xi_b$  on  $H$  it is given by the determinant  $\det\left(\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}\right)$ , where  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  is the homogeneous cubic polynomial associated with  $F$ .

**PROPOSITION 16.** *Let  $F$  be a symmetric trilinear form whose Hessian vanishes identically. Then  $F$  is not realizable as cup-form of a Kählerian 3-fold.*

*Proof.* Let  $X$  be a complex 3-fold with a Kähler metric  $g$ . The Kähler class  $[\omega_g] \in H^2(X, \mathbf{R})$  defines a multiplication map  $\cdot [\omega_g]: H^2(X, \mathbf{R}) \rightarrow H^4(X, \mathbf{R})$ , which is an isomorphism by the Hard Lefschetz Theorem [G/H]. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

**COROLLARY 6.** *Cubic forms  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  which depend on strictly less than  $b = \text{rk}_{\mathbf{Z}} H$  variables are not realizable as cup-forms of Kählerian 3-folds with  $b_2 = b$ .*

By considering the Hessian of a cup-form over the reals one obtains further conditions.

**DEFINITION 4.** *Let  $F \in S^3 H^\vee$  be a symmetric trilinear form on a free  $\mathbf{Z}$ -module of rank  $b$ .*

*The Hesse cone of  $F$  is the subset  $\mathcal{H}_F \subset H_{\mathbf{R}}$  defined by  $\mathcal{H}_F := \{h \in H_{\mathbf{R}} \mid (-1)^b \det(F^t(h)) < 0\}$ .*

The index cone  $\mathcal{J}_F$  of  $F$  is the subset  $\mathcal{J}_F := \{h \in \mathcal{H}_F \mid F^t(h) \in S^2 H_{\mathbf{R}}^\vee\}$  has signature  $(1, -1, \dots, -1)$ .

Clearly  $\mathcal{J}_F$  is an open subcone of  $\mathcal{H}_F$  which coincides with  $\mathcal{H}_F$  iff  $b \leq 2$ .

**THEOREM 5.** *Let  $F_X \in S^3 H^2(X, \mathbf{Z})^\vee$  be the cup-form of a smooth projective 3-fold with  $h^{0,2}(X) = 0$ . Then  $F_X$  has a non-empty index cone.*

*Proof.* Let  $h \in H^2(X, \mathbf{Z})$  be the dual class of a hyperplane section  $Y$  in some projective embedding. The inclusion  $i: Y \hookrightarrow X$  induces a monomorphism  $i^*: H^2(X, \mathbf{Z}) \rightarrow H^2(Y, \mathbf{Z})$  by the weak Lefschetz theorem. The symmetric bilinear form  $F_X^t(h) \in S^2 H^2(X, \mathbf{Z})^\vee$  is simply the pull-back of the cup-form of  $Y$  under the inclusion  $i^*$ ; it is therefore non-degenerate by the Hard Lefschetz theorem [L]. Applying the Hodge index theorem to  $Y$  we see that the real bilinear form  $F_X^t(h) \in S^2 H^2(X, \mathbf{R})^\vee$  must have one positive and  $b - 1$  negative eigenvalues. In other words:  $h \in I_{F_X}$ .

**REMARK 13.** This result has two applications: it provides topological ‘upper bounds’ for the ample cone of a projective 3-fold with  $h^{0,2} = 0$ , and it gives further restrictions on symmetric trilinear forms to be realizable as cup-forms of projective 3-folds with  $h^{0,2} = 0$  if  $b \geq 4$ .

These applications will be discussed in section 5.

We will now describe examples of 1-connected, non-Kählerian, complex 3-folds and determine their topological structure.

**EXAMPLE 10 (Calabi-Eckmann).** E. Calabi and B. Eckmann have defined complex structures  $X_\tau$ , depending on a parameter  $\tau$ , on the product  $S^3 \times S^3[C/E]$ . Their manifolds are principal fiber bundles over  $\mathbf{P}^1 \times \mathbf{P}^1$  whose fiber and structure group is the elliptic curve  $E_\tau = \mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$ ,  $\text{Im}(\tau) > 0$ .

The Calabi-Eckmann manifolds are homogeneous, non-Kählerian 3-folds of algebraic dimension 2.

**EXAMPLE 11 (Maeda).** H. Maeda has generalized the Calabi-Eckmann construction. He constructed fiber bundles  $X'_\tau$  over Hirzebruch surfaces  $\mathbf{F}_n$ ,  $n \geq 0$ , whose fiber and structure group are an elliptic curve  $E_\tau$  and  $\text{Aut}(E_\tau)$  respectively [M].  $X'_\tau$  is again diffeomorphic to  $S^3 \times S^3$ , and therefore non-Kählerian. Maeda’s manifolds  $X'_\tau$  are homogeneous if and only if  $n = 0$  in which case they are Calabi-Eckmann 3-folds.

The Calabi-Eckmann construction can also be generalized in the following way:

Let  $S^2 \tilde{\times} S^4$  be the non-trivial  $S^4$ -bundle over  $S^2$ , i.e.  $S^2 \tilde{\times} S^4$  is the unique 1-connected, closed, oriented, differentiable 6-manifold with  $H_2(S^2 \tilde{\times} S^4, \mathbf{Z}) \cong \mathbf{Z}$  and  $b_3 = 0$ , whose cup-form and Pontrjagin class vanish, but whose Stiefel-Whitney class  $w_2$  is non-zero.

**THEOREM 6.** *For any integer  $b \geq 0$  there exist compact complex 3-folds  $X_b$ , and  $X_b^-$  if  $b \geq 1$ , which are homeomorphic to  $\#_b S^2 \times S^4 \#_{b+1} S^3 \times S^3$ , and  $S^2 \tilde{\times} S^4 \#_{b-1} S^2 \times S^4 \#_{b+1} S^3 \times S^3$ .*

*Proof.* Let  $Y$  be a 1-connected, compact complex surface with  $p_g(Y) = 0$  and  $b_2(Y) \geq 2$ , and let  $E = \mathbf{C}/\Gamma$  be the elliptic curve associated to the lattice  $\Gamma \subset \mathbf{C}$ . We want to construct the required 3-folds as total spaces of principal  $E$ -bundles over  $Y$ . Let  $c: H_2(Y, \mathbf{Z}) \rightarrow \Gamma$  be an arbitrary epimorphism. The corresponding cohomology class  $c \in H^2(Y, \Gamma)$  defines a topological principal bundle over  $Y$  with fiber and structure group  $E = \mathbf{C}/\Gamma$  as follows immediately from the identification of the classifying space  $BE \simeq K(\Gamma, 2)$ .

Let  $\mathcal{O}_Y(E)$  be the sheaf of germs of holomorphic maps from  $Y$  to  $E$ . We have a short exact sequence  $0 \rightarrow \Gamma \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y(E) \rightarrow 0$  and a corresponding exact cohomology sequence

$$\rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{O}_Y(E)) \xrightarrow{\delta} H^2(Y, \Gamma) \rightarrow H^2(Y, \mathcal{O}_Y) \rightarrow$$

By our assumptions  $\delta$  is an isomorphism, so that every topological principal  $E$ -bundle admits a holomorphic structure. Let  $X$  be the total space of such a bundle corresponding to a surjective map  $c: H_2(Y, \mathbf{Z}) \rightarrow \Gamma$ . The homotopy sequence of the fibration  $p: X \rightarrow Y$  yields the sequence

$$0 \rightarrow \pi_2(X) \xrightarrow{p^*} \pi_2(Y) \rightarrow \pi_1(E) \rightarrow \pi_1(X) \xrightarrow{p^*} \pi_1(Y) \rightarrow 0.$$

Since  $Y$  is 1-connected,  $\pi_2(Y)$  can be identified with  $H_2(Y, \mathbf{Z})$ , and then the boundary map  $\pi_2(Y) \rightarrow \pi_1(E)$  becomes the characteristic map  $c: H_2(Y, \mathbf{Z}) \rightarrow \Gamma$  of the bundle. This implies  $\pi_1(X) = \{1\}$ , whereas  $H_2(X, \mathbf{Z})$  is given by:  $0 \rightarrow H_2(X, \mathbf{Z}) \xrightarrow{p^*} H_2(Y, \mathbf{Z}) \xrightarrow{c} \Gamma \rightarrow 0$ .

In particular,  $H_2(X, \mathbf{Z})$  is free as a submodule of  $H_2(Y, \mathbf{Z})$ , and by dualizing the last sequence we obtain an identification (via  $p^*$ )

$$H^2(X, \mathbf{Z}) = H^2(Y, \mathbf{Z})/\Gamma^\vee.$$

The cup-form  $F_X$  of  $X$  is therefore trivial. In order to calculate  $p_1(X)$  and  $w_2(X)$ , we use the exact sequence of tangent sheaves:  $0 \rightarrow T_{X/Y} \rightarrow T_X$

$\rightarrow p^*T_Y \rightarrow 0$ . Since  $T_{X/Y}$  is a trivial bundle, the characteristic classes of  $X$  are simply the pullbacks of the corresponding classes of  $Y$ . But the map  $p^*: H^4(Y, \mathbf{Z}) \rightarrow H^4(X, \mathbf{Z})$  is zero, since  $\langle p^*(\varepsilon) \cup p^*(\alpha), [X] \rangle = \langle \varepsilon \cup \alpha, p_*[X] \rangle = 0$  for all classes  $\varepsilon \in H^4(Y, \mathbf{Z})$ , and  $\alpha \in H^2(Y, \mathbf{Z})$ .

Thus  $p_1(X) = 0$ , and  $w_2(X)$  is the residue of  $w_2(Y) \in H^2(Y, \mathbf{Z}_{/2})$  modulo  $\Gamma^\vee / {}_2\Gamma^\vee$ .

The Euler characteristic of  $X$  is zero, so that from  $b_2(X) = b_2(Y) - 2$  we find  $b_3(X) = 2(b_2(Y) - 1)$ . The system of invariants associated to the manifold  $X$  is therefore given by

$$(b_2(Y) - 1, H^2(Y, \mathbf{Z}) / \Gamma^\vee, w_2(Y) \pmod{\Gamma^\vee / {}_2\Gamma^\vee}, 0, 0, 0),$$

i.e.  $X$  is diffeomorphic to

$$\#_{b_2(Y)-2} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3 \text{ if } w_2(Y) \in \Gamma^\vee / {}_2\Gamma^\vee,$$

and to  $S^2 \tilde{\times} S^4 \#_{b_2(Y)-3} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3$  if  $b_2(Y) \geq 3$ , and  $w_2(Y) \notin \Gamma^\vee / {}_2\Gamma^\vee$ .

EXAMPLE 12 (Kato). In the two papers [K1], [K2] M. Kato studies the class of compact, complex 3-folds  $X$  containing smooth rational curves with neighborhoods biholomorphic to those of projective lines in  $\mathbf{P}^3$ . On this class of 3-folds, called class  $L$ , he defines a semi-group structure  $+$  with neutral element  $\mathbf{P}^3$ .

Kato's connecting operation  $+$  is defined by removing 'lines'  $L_i \subset X_i$  from 3-folds  $X_i, i = 1, 2$ , and by identifying the complements  $X_i \setminus L_i$  along open sets  $U_i \setminus L_i$  obtained from suitable neighborhoods  $U_i \subset X_i$ .

Starting with a certain elliptic fiber space  $X_1$  over the blow-up of  $\mathbf{P}^1 \times \mathbf{P}^1$  in a point, he constructs a sequence of 3-folds  $X_n := X_1 + X_{n-1}, n \geq 2$ . The 3-folds  $X_n$  are 1-connected spin-manifolds with  $H_2(X_n, \mathbf{Z}) = \mathbf{Z}$ . Their cup-forms  $F_{X_n}$ , and their Pontrjagin classes  $p_1(X_n)$  are in terms of a (normalized) generator  $e_n \in H^2(X_n, \mathbf{Z})$  and its dual class  $\varepsilon_n \in H^4(X_n, \mathbf{Z})$  given by  $F_{X_n}(xe_n) = (n-1)x^3$ , and  $p_1(X_n) = 4(n-1)\varepsilon_n$  ( $\varepsilon_n(e_n) = 1$ ). The third Betti-number of  $X_n$  is  $4n$ .

In particular,  $X_1$  is diffeomorphic to  $S^2 \times S^4 \#_2 S^3 \times S^3$ , and  $X_2$  is diffeomorphic to  $\mathbf{P}^3 \#_4 S^3 \times S^3$ . It is interesting to note that the Chern-numbers  $c_1^3, c_1 c_2$  of the  $X_n$  are  $c_1^3 = 64(1-n)$ ,  $c_1 c_2 = 24(1-n)$ , i.e. they satisfy  $8c_1 c_2 = 3c_1^3$ . For projective manifolds of general type this equality is characteristic for ball quotients [Y].

EXAMPLE 13 (Twistor spaces). Let  $p: Z \rightarrow M$  be the twistor fibration of a closed, oriented Riemannian 4-manifold  $(M, g)$ .  $Z$  carries a natural almost complex structure which is integrable if and only if  $g$  is self-dual [A/H/S].

Examples of 1-connected 4-manifolds which admit self-dual structures are  $S^4$ ,  $\#_n \mathbf{P}^2$ , and  $K3$ -surfaces.

The total spaces of their twistor fibrations are 1-connected complex 3-folds which may be Moishezon for  $S^4$  and  $\#_n \mathbf{P}^2$  [C], but which are usually non-Kähler [Hi]. We leave it to the reader to calculate the topological invariants of these 3-folds. There is an interesting relation between Twistor spaces of connected sums and Kato's connection operation  $+$  for class  $L$  manifolds [K2], [D/F].

EXAMPLE 14 (Oguiso). In a recent preprint [O1] K. Oguiso constructs examples of 1-connected, Moishezon Calabi-Yau 3-folds with very interesting cup-forms. He proves that for every integer  $d \geq 1$  there exists a smooth complete intersection  $X'_d$  of type  $(2, 4)$  in  $\mathbf{P}^5$  which contains a non-singular rational curve  $C_d$  of degree  $d$  with normal bundle  $N_{C_d/X_d} = \mathcal{O}_{C_d}(-1)^{\oplus 2}$ .

The 3-fold  $X'_d$  can now be flopped along  $C_d$ , i.e.  $C_d$  can be blown up to  $\mathbf{P}(N_{C_d/X_d}) \cong \mathbf{P}^1 \times \mathbf{P}^1$ , and then 'blown down in the other direction'. The resulting 3-fold  $X_d$  is a 1-connected Moishezon manifold with trivial canonical bundle and cup-form  $F_{X_d}$  given by  $F_{X_d}(xe_d) = (d^3 - 8)x^3$ . Here  $e_d \in H^2(X_d, \mathbf{Z})$  is the normalized generator corresponding to the strict transform of the negative of a hyperplane section of  $X'_d$ . The Pontrjagin class of  $X_d$  is  $p_1(X_d) = (112 + 4d)\varepsilon_d$  where  $\varepsilon_d \in H^4(X_d, \mathbf{Z})$  denotes the generator with  $\varepsilon_d(e_d) = 1$ . Since the Euler-number does not change under a flop we have  $b_3(X_d) = 180$  for every  $d$ .

## 5. COMPLEX 3-FOLDS WITH SMALL $b_2$

In this section we investigate the following natural problem: Which cubic forms can be realized as cup-forms of compact complex 3-folds? For small  $b_2$  something can be said: Any core of a 1-connected, closed, oriented differentiable 6-manifold with  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$  is homotopy equivalent to the core of a 1-connected complex 3-fold. In the case  $b_2 = 2$ , at least every discriminant  $\Delta$  is realizable by a complex manifold. If  $b_2 = 3$  we can realize all types of complex cubics with one exception, the union of a smooth conic and a tangent line. In addition to these realization results we prove a finiteness