

## 2.2 Homotopy types with a given cohomology ring

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*Proof.*  $W_o$  is characteristic for  $F$  if and only if  $q_{\bar{F}} = \bar{F}^t(W_o)$ .

In terms of a  $\mathbf{Z}$ -basis  $\{e_1, \dots, e_b\}$  for  $H$  the condition  $q_{\bar{F}} \in \text{Im}(\bar{F}^t)$  translates into a simple rank condition over  $\mathbf{Z}_{/2}$ : the  $\mathbf{Z}_{/2}$ -rank of the  $b \times \binom{b+1}{2}$ -matrix  $A$  representing  $\bar{F}^t$  must be equal to the  $\mathbf{Z}_{/2}$ -rank of the matrix  $A$  extended by the column  $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leq i < j \leq b}$

EXAMPLE 3. Let  $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$  be free of rank 2,  $F \in S^3 H^\vee$  given by  $e_1^3 = a, e_1^2 e_2 = b, e_1 e_2^2 = c, e_2^3 = d$  with  $a, b, c, d \in \mathbf{Z}$ . The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

## 2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr’s classification theorem we know that in algebraic terms this means the following: fix a non-negative integer  $r_o$ , a finitely generated free abelian group  $H_o$ , and a symmetric trilinear form  $F_o \in S^3 H_o^\vee$  which admits characteristic elements.

Let  $\mathcal{M}(r_o, H_o, F_o)$  be the set of 1-connected, closed, oriented, 6-dimensional manifolds  $X$  with  $b_3(X) = 2r_o$ , such that there exists an isomorphism  $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$  with  $\alpha^* F_X = F_o$ . Denote by  $\text{Aut}(F_o)$  the subgroup of  $\mathbf{Z}$ -automorphisms of  $H_o$  which leave  $F_o \in S^3 H_o^\vee$  invariant;  $\text{Aut}(F_o)$  acts on pairs  $(w, [l]) \in \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$  in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^* [l]) .$$

Let  $\text{Aut}(F_o) \backslash \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$  be the set of  $\text{Aut}(F_o)$ -orbits.

A manifold  $X$  in  $\mathcal{M}(r_o, H_o, F_o)$  and an isomorphism  $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$  with  $\alpha^* F_X = F_o$  yields a well-defined  $\text{Aut}(F_o)$ -orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^* [p_1(X) + 24T]) \text{ (modulo } \text{Aut}(F_o) \text{) ,}$$

where  $T \in H^4(X, \mathbf{Z})$  is an arbitrary integral lifting of  $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$ .

The set of oriented homotopy types  $\mathcal{M}(r_o, H_o, F_o) / \simeq$  of manifolds in  $\mathcal{M}(r_o, H_o, F_o)$  can now be described in the following way:

PROPOSITION 3. *The assignment  $X \mapsto (\alpha^{-1}(w_2(X)), \alpha^*[p_1(X) + 24T])$  (modulo  $\text{Aut}(F_o)$ ) defines an injection.*

$$I: \mathcal{M}(r_o, H_o, F_o) / \cong \rightarrow_{\text{Aut}(F_o)} \backslash \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}.$$

*Proof.* Suppose  $X$  and  $X'$  are manifolds in  $\mathcal{M}(r_o, H_o, F_o)$ ,  $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$  and  $\alpha': H_o \rightarrow H^2(X', \mathbf{Z})$  isomorphisms with  $\alpha^*F_X = F_o$  and  $(\alpha')^*F_{X'} = F_o$ .  $X$  and  $X'$  have the same image under  $I$  iff there exists an automorphism  $\gamma \in \text{Aut}(F_o)$  with  $\gamma\alpha^{-1}(w_2(X)) = (\alpha')^{-1}w_2(X')$  and  $(\gamma^{-1})^*\alpha^*[p_1(X) + 24T] = (\alpha')^*[p_1(X') + 24T']$ . Consider  $\beta := \alpha \circ \gamma \circ \alpha^{-1}: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$ ;  $\beta$  is obviously an isomorphism with  $\beta^*F_{X'} = F_X$ ,  $\beta w_2(X) = w_2(X')$ , and  $\beta^*[p_1(X') + 24T'] = [p_1(X) + 24T]$ ; but this means that the systems of invariants associated with  $X$  and  $X'$  are weakly equivalent, and therefore  $X$  and  $X'$  oriented homotopy equivalent.

A complete description of the set  $\mathcal{M}(r_o, H_o, F_o) / \cong$  i.e. of the image of  $I$  is only possible if the automorphism group  $\text{Aut}(F_o)$  is known; this can be a serious problem, but we will see that the ‘general’ automorphism group is finite (and usually small), so that the next proposition gives a reasonable estimate for the number of elements in  $\mathcal{M}(r_o, H_o, F_o) / \cong$ .

PROPOSITION 4. *Fix  $r_o \in \mathbf{N}$ , a finitely generated free abelian group  $H_o$ , and a symmetric trilinear form  $F_o \in S^3H_o^\vee$  which admits characteristic elements. Set  $b := rk_{\mathbf{Z}}H_o$ ,  $s := rk_{\mathbf{Z}/2}(\bar{F}_o^t)$ , and let  $t := rk_{\mathbf{Z}/2}(\cdot_{\bar{F}_o})$  be the  $\mathbf{Z}/2$ -rank of the  $\mathbf{Z}/2$ -linear square map  $\cdot_{\bar{F}_o}: \bar{H}_o \rightarrow \bar{H}_o^\vee$  sending  $\bar{u} \in \bar{H}_o$  to  $\bar{u}^2 \in \bar{H}_o^\vee$ . Then  $\mathcal{M}(r_o, H_o, F_o) / \cong$  contains at most  $2^{2b-s-t}$  elements.*

*Proof.* Fix any admissible system of invariants  $(r_o, H_o, w_o, \tau_o, F_o, p_o)$  for a manifold in  $\mathcal{M}(r_o, H_o, F_o)$ . Given  $(r_o, H_o, F_o)$ , we know from the last lemma that the possible elements  $w_o$  form a coset of  $\text{Ker}(\bar{F}_o^t)$  in  $\bar{H}_o$ , so that there exist precisely  $2^{b-s}$  such elements. It remains to count the classes  $[l] \in H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ , such that the  $\text{Aut}(F_o)$ -orbit of  $(w_o, [p_o + 24T_o + l])$  lies in the image of  $I$ .

To understand the latter condition we fix integral liftings  $W_o, \in H_o, T_o \in H_o^\vee$  of  $w_o$  and  $\tau_o$  satisfying the admissibility conditions

- i)  $W_o^3 \equiv (p_o + 24T_o)(W_o) \pmod{48}$
- ii)  $p_o(x) \equiv 4x^3 + 6x^2W_o + 3xW_o^2 \pmod{24} \quad \forall x \in H_o.$

Clearly the  $\text{Aut}(F_o)$ -orbit of  $(w_o, [p_o + 24T_o + l])$  lies in the image of  $I$  if and only if

i')  $W_o^3 \equiv (p_o + 24T_o + l)(W_o) \pmod{48}$ ,

ii')  $(p_o + l)(x) \equiv 4x^3 + 6x^2W_o + 3xW_o^2 \pmod{24} \quad \forall x \in H_o$ ,

which is equivalent to  $l(W_o) \equiv 0 \pmod{48}$ , and  $l \equiv 0 \pmod{24H_o^\vee}$  because of i) and ii).

Now, by definition of the subgroup  $U_{F_o} \subset H_o^\vee / 48H_o^\vee$  we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & \text{Ker}(\cdot \bar{F}_o) & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow \text{Ker}(24 \cdot \bar{F}_o) & \hookrightarrow & H_o / 2H_o & \xrightarrow{24 \cdot \bar{F}_o} & U_{F_o} & \rightarrow & 0 \\
 & & \cdot \bar{F}_o \downarrow & & \downarrow & & \\
 0 \rightarrow & H_o^\vee / 2H_o^\vee & \xrightarrow{24} & H_o^\vee / 48H_o^\vee & \rightarrow & H_o^\vee / 24H_o^\vee \rightarrow & 0 \\
 & \downarrow & & \downarrow & & \parallel & \\
 0 \rightarrow & \text{Coker}(\cdot \bar{F}_o) & \rightarrow & H_o^\vee / 48H_o^\vee / U_{F_o} & \rightarrow & H_o^\vee / 24H_o^\vee \rightarrow & 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & & 
 \end{array}$$

The number of elements  $[l] \in H_o^\vee / 48H_o^\vee / U_{F_o}$  to be counted coincides therefore with the cardinality of the kernel of the map  $ev(w_o): \text{Coker}(\cdot \bar{F}_o) \rightarrow \mathbf{Z}_{/2}$  induced by evaluation in  $w_o$ . This number is at most  $2^{b-t}(2^{b-t-1}$  if  $w_o \neq 0$  and  $t \neq b$ ).

**COROLLARY 2.** *If the  $\mathbf{Z}_{/2}$ -rank  $s = rk_{\mathbf{Z}_{/2}}(\cdot \bar{F}_o)$  is maximal, then  $\mathcal{M}(r_o, H_o, F_o) / \cong$  contains at most one class.*

*Proof.* Suppose  $\cdot \bar{F}_o: \bar{H}_o \rightarrow \bar{H}_o^\vee$  is surjective; then  $\bar{F}_o^t: \bar{H}_o \rightarrow S^2 \bar{H}_o^\vee$  must have a trivial kernel, since  $\bar{h}\bar{x}^2 = 0$  for all  $\bar{x} \in \bar{H}_o$  implies  $\bar{h} = 0$  if every linear form is a square. But this means  $s = t = b$ , so that  $\mathcal{M}(r_o, H_o, F_o) / \cong$  has at most one element.

**EXAMPLE 4.** Let  $H_o = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ ,  $e_1^3 = a$ ,  $e_1^2e_2 = b$ ,  $e_1e_2^2 = c$ ,  $e_2^3 = d$ . If  $\bar{b} \equiv \bar{c} \pmod{2}$ , and  $\bar{a}\bar{d} - \bar{b}\bar{c} \equiv 1 \pmod{2}$ , then  $\mathcal{M}(r_o, H_o, F_o) / \cong$  contains precisely one class for every  $r_o \geq 0$ .