

## 2.1 COHOMOLOGY RINGS OF 6-MANIFOLDS

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## 2.1 COHOMOLOGY RINGS OF 6-MANIFOLDS

Let  $(r, H, w, \tau, F, p)$  be a system of invariants as in section 1; recall that it is admissible iff for every  $W \in H$ ,  $T \in H^\vee$  with  $\bar{W} = w(\text{mod } 2)$ ,  $\bar{T} \equiv \tau(\text{mod } 2)$  the following congruence holds:

$$(*) \quad W^3 \equiv (p + 24T)(W) \pmod{48}.$$

LEMMA 1.  $(r, H, w, \tau, F, p)$  is admissible if and only if there exist  $W_o \in H$ ,  $T_o \in H^\vee$  with  $\bar{W}_o \equiv w(\text{mod } 2)$ ,  $\bar{T}_o \equiv \tau(\text{mod } 2)$ , such that

- i)  $W_o^3 \equiv (p + 24T_o)(W_o) \pmod{48}$
- ii)  $p(x) \equiv 4x^3 + 6x^2 W_o + 3x W_o^2 \pmod{24} \quad \forall x \in H$ .

*Proof.* Obvious since the set of integral lifts of  $w$  is a coset  $W_o + 2H$ .

DEFINITION 3. Let  $F \in S^3 H^\vee$  be a symmetric trilinear form on a finitely generated free abelian group  $H$ . An element  $W \in H$  is characteristic for  $F$  iff

$$(**) \quad x \cdot y \cdot (x + y + W) \equiv 0 \pmod{2} \quad \forall x, y \in H.$$

LEMMA 2.  $W \in H$  is a characteristic element for  $F \in S^3 H^\vee$  if and only if the function  $l_W: H \rightarrow \mathbf{Z}$ ,  $l_W(x) := 4x^3 + 6x^2 W + 3x W^2$  is linear in  $x$  modulo 24.

*Proof.*  $l_W(x + y) = l_W(x) + l_W(y) + 12(x^2 y + x y^2 + x y W)$ , whence the assertion.

The existence of characteristic elements is a necessary and sufficient condition for a cubic form  $F \in S^3 H^\vee$  to be realizable by a manifold. In fact, we have:

PROPOSITION 2. A given cubic form  $F \in S^3 H^\vee$  on a finitely generated free abelian group  $H$  is realizable as cup-form of a 1-connected, closed, oriented, 6-dimensional manifold with torsion-free homology if and only if it possesses a characteristic element.

*Proof.* If  $(r, H, w, \tau, F, p)$  is an admissible system of invariants, and  $W_o \in H$  any integral lift of  $w$ , then we have  $p(x) \equiv 4x^3 + 6x^2 W_o + 3x W_o^2 \pmod{24} \quad \forall x \in H$ , i.e. the function  $l_{W_o}: H \rightarrow \mathbf{Z}$  is linear modulo 24, and  $W_o$  is therefore characteristic for  $F$ . Conversely, suppose  $W_o \in H$  is a characteristic element for a cubic form  $F \in S^3 H^\vee$ ; let  $w := \bar{W}_o(\text{mod } 2)$ ,  $r := 0$ .

By the main lemma we have to construct linear forms  $p, T \in H^\vee$ , such that

- i)  $W_\circ^3 \equiv (p + 24T)(W_\circ) \pmod{48}$
- ii)  $p(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24} \quad \forall x \in H$ .

The function  $l_{W_\circ}: H \rightarrow \mathbf{Z}, l_{W_\circ}(x) = 4x^3 + 6x^2 W_\circ + 3x W_\circ^2$  is linear modulo 24 since  $W_\circ$  is a characteristic element for  $F$ : we therefore choose a linear form  $p_\circ \in H^\vee$  with  $p_\circ(x) \equiv l_{W_\circ}(x) \pmod{24} \quad \forall x \in H$ . Substituting  $x = W_\circ$  we find  $p_\circ(W_\circ) \equiv 13 W_\circ^3 \pmod{24}$ ; but since  $W_\circ$  is characteristic we have  $W_\circ^3 \equiv 0 \pmod{2}$ , thus  $p_\circ(W_\circ) \equiv W_\circ^3 \pmod{24}$ . Write  $p_\circ(W_\circ) = W_\circ^3 + 24k$  for some  $k \in \mathbf{Z}$ .

case 1)  $k \equiv 0 \pmod{2}$ : define  $p := p_\circ, T := 0$ .

case 2)  $k \equiv 1 \pmod{2}$ : we must find a linear form  $T_\circ \in H^\vee$  with  $T_\circ(W_\circ) \equiv 1 \pmod{2}$ ; clearly this can be done if and only if  $W_\circ$  is not divisible by 2. If  $W_\circ$  were divisible by 2,  $W_\circ = 2V_\circ$  for some  $V_\circ \in H$ , then  $2p_\circ(V_\circ) = p_\circ(W_\circ) = W_\circ^3 + 24k = 8V_\circ^3 + 24k$  would give  $p_\circ(V_\circ) = 4V_\circ^3 + 12k$ ; then, using  $p_\circ(V_\circ) \equiv 4V_\circ^3 + 6V_\circ^2 W_\circ + 3V_\circ W_\circ^2 \pmod{24}$  we would find  $k \equiv 0 \pmod{2}$ , which is not the case by assumption.

This shows that  $F \in S^3 H^\vee$  is realizable by a topological manifold with Pontrjagin class  $p_\circ$  and non-vanishing triangulation obstruction  $\tau_\circ := \bar{T}_\circ \pmod{2}$ . In order to realize  $F$  by a smooth manifold, one can take  $p := p_\circ + 24T_\circ$ , and  $\tau := 0$ .

REMARK 3. The topological counterpart of the existence of a characteristic element for a given cubic form  $F \in S^3 H^\vee$  is the existence of a mod-2 Steenrod-algebra structure, which is a necessary condition for a ring to be a cohomology ring.

The existence and the classification of characteristic elements for a given cubic form is essentially a linear algebra problem over  $\mathbf{Z}_{/2}$ . To see this, let  $F \in S^3 H^\vee$  be a fixed cubic form on a finitely generated free abelian group  $H$ . Associated with  $F$  we have a linear map  $F^t: H \rightarrow S^2 H^\vee$  sending an element  $h \in H$  to the bilinear form  $F^t(h): H \otimes H \rightarrow \mathbf{Z}, (x, y) \rightarrow x \cdot y \cdot h$ . Let  $\bar{H} := H/_{2H}$ .  $\bar{F} \in S^3 \bar{H}^\vee$  be the reductions of  $H$  and  $F$  modulo 2, and let  $-: H \rightarrow \bar{H}$  be the natural epimorphism. The symmetric trilinear form  $\bar{F}$  on the  $\mathbf{Z}_{/2}$ -module  $\bar{H}$  defines a natural symmetric bilinear form  $q_{\bar{F}} \in S^2 \bar{H}^\vee$  given by  $q_{\bar{F}}(\bar{x}, \bar{y}) := \bar{x} \cdot \bar{y} \cdot (\bar{x} + \bar{y})$ .

LEMMA 3.  $F \in S^3 H^\vee$  admits characteristic elements if and only if  $q_{\bar{F}}$  lies in the image of  $\bar{F}^t \in \text{Hom}_{\mathbf{Z}}(H, S^2 \bar{H}^\vee)$ . The set of all characteristic elements for  $F$  is a coset of the form  $W_\circ + \text{Ker}(\bar{F}^t)$ .

*Proof.*  $W_o$  is characteristic for  $F$  if and only if  $q_{\bar{F}} = \bar{F}^t(W_o)$ .

In terms of a  $\mathbf{Z}$ -basis  $\{e_1, \dots, e_b\}$  for  $H$  the condition  $q_{\bar{F}} \in \text{Im}(\bar{F}^t)$  translates into a simple rank condition over  $\mathbf{Z}_{/2}$ : the  $\mathbf{Z}_{/2}$ -rank of the  $b \times \binom{b+1}{2}$ -matrix  $A$  representing  $\bar{F}^t$  must be equal to the  $\mathbf{Z}_{/2}$ -rank of the matrix  $A$  extended by the column  $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leq i \leq j \leq b}$

EXAMPLE 3. Let  $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$  be free of rank 2,  $F \in S^3 H^\vee$  given by  $e_1^3 = a$ ,  $e_1^2 e_2 = b$ ,  $e_1 e_2^2 = c$ ,  $e_2^3 = d$  with  $a, b, c, d \in \mathbf{Z}$ . The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

## 2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr's classification theorem we know that in algebraic terms this means the following: fix a non-negative integer  $r_o$ , a finitely generated free abelian group  $H_o$ , and a symmetric trilinear form  $F_o \in S^3 H_o^\vee$  which admits characteristic elements.

Let  $\mathcal{M}(r_o, H_o, F_o)$  be the set of 1-connected, closed, oriented, 6-dimensional manifolds  $X$  with  $b_3(X) = 2r_o$ , such that there exists an isomorphism  $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$  with  $\alpha^* F_X = F_o$ . Denote by  $\text{Aut}(F_o)$  the subgroup of  $\mathbf{Z}$ -automorphisms of  $H_o$  which leave  $F_o \in S^3 H_o^\vee$  invariant;  $\text{Aut}(F_o)$  acts on pairs  $(w, [l]) \in \bar{H}_o \times H_o^\vee / {}_{48}H_o^\vee / {}_{U_{F_o}}$  in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^*[l]) .$$

Let  ${}_{\text{Aut}(F_o)} \backslash \bar{H}_o \times H_o^\vee / {}_{48}H_o^\vee / {}_{U_{F_o}}$  be the set of  $\text{Aut}(F_o)$ -orbits.

A manifold  $X$  in  $\mathcal{M}(r_o, H_o, F_o)$  and an isomorphism  $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$  with  $\alpha^* F_X = F_o$  yields a well-defined  $\text{Aut}(F_o)$ -orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^*[p_1(X) + 24T]) \text{ (modulo } \text{Aut}(F_o) \text{)},$$

where  $T \in H^4(X, \mathbf{Z})$  is an arbitrary integral lifting of  $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$ .

The set of oriented homotopy types  $\mathcal{M}(r_o, H_o, F_o) / \simeq$  of manifolds in  $\mathcal{M}(r_o, H_o, F_o)$  can now be described in the following way: