

## 5. The global form of the Stokes formula on $C^1$ manifolds

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as  $\varphi(Q) = 0$  for any  $Q$  of the types (1)-(2) described above, and since  $\varphi$  is additive, it follows that  $\psi(Q'_i) = \varphi(Q_i)$  for any  $i \in J$ . In particular,

$$\left| \sum_{i \in J} \psi(Q'_i) \right| = |\varphi(P)| \leq \left| \int_{\overset{\circ}{\Omega} \cap \partial P} u \right| + \left| \int_{\overset{\circ}{P} \cap b\Omega} u \right| + \left| \iint_{P \cap \Omega} f \right|.$$

By (4.2), the uniformly local  $(n-1)$ -integrability of  $u$ , the integrability of  $u$  on  $b\Omega$  and the integrability of  $f$  on  $\Omega$ , the right hand side of the above equality can be made arbitrarily small, provided  $\sum_{i \in J} \mu_{n-1}(Q'_i)$  is sufficiently small. However, since  $A'$  has Lebesgue measure zero in  $\mathbf{R}^{n-1}$ , this can be readily taken care of and this completes the proof of the theorem.  $\square$

REMARK 4.3. As an inspection of the proofs shows, Theorem 4.1 and Lemma 4.2 continue to hold in the case when the locally  $(n-1)$ -integrable form  $u$  is *uniformly*  $(n-1)$ -integrable only in a small neighborhood of  $\mathcal{S}(u)$ .

## 5. THE GLOBAL FORM OF THE STOKES FORMULA ON $C^1$ MANIFOLDS

In this section we shall present a coordinate free version of the main result of section 4. Throughout this section, we let  $M$  be a fixed, oriented, Hausdorff, differentiable manifold of class  $C^1$ , and real dimension  $n$ .

DEFINITION 5.1. *A subset  $\Omega$  of  $M$  is called a  $C^1$  domain if for any  $a \in \Omega \setminus \overset{\circ}{\Omega}$ , there exist an open neighborhood  $U$  of  $a$  in  $M$  and a  $C^1$  diffeomorphism  $f = (f_1, f_2, \dots, f_n)$  of  $U$  onto an open neighborhood  $V$  of the origin in  $\mathbf{R}^n$ , such that*

$$U \cap \Omega = \{x \in U; f_n(x) \leq 0\}.$$

Clearly, the *border of the domain*  $\Omega$ ,  $b\Omega := \Omega \setminus \overset{\circ}{\Omega}$  is either the empty set or a  $(n-1)$ -dimensional  $C^1$ -submanifold of  $M$  assumed with the standard induced orientation. Note that a simple application of the implicit function theorem shows that any  $C^1$  domain is also a Lipschitz domain in  $\mathbf{R}^n$ .

It is not difficult to see that the class of Lipschitz domains described in Definition 1.1 is not invariant under the action of bi-Lipschitz diffeomorphisms of  $\mathbf{R}^n$ . In particular, Theorem 4.1 cannot be reformulated invariantly. To remedy this, for the rest of this section we shall slightly adjust

our previous definitions to the  $C^1$  framework by carrying out the following simple modification. That is, whenever applicable, we shall replace “Lipschitz embedding” by “ $C^1$ -embedding”, i.e. Lipschitz embeddings which are  $C^1$  functions. Note that, in particular, the condition (1.1) is in this case equivalent with the continuity of the functions

$$\frac{\partial h(s, x)}{\partial x_i} : S \times \omega \rightarrow \Omega, \quad i = 1, 2, \dots, n-1.$$

Assuming this modification, all the previously introduced notions become invariant to  $C^1$  diffeomorphisms and, hence, meaningful on  $C^1$  manifolds. More specifically, we make the following.

**DEFINITION 5.2.** *Let  $\Omega$  be a  $C^1$  domain of  $M$ . A  $(n-1)$ -form  $u$  is said to be absolutely continuous (uniformly  $(n-1)$ -locally integrable) on  $\Omega$  if for any point  $P \in \Omega$  there exists a local coordinate map  $h: U \rightarrow \mathbf{R}^n$  of  $M$  with  $P \in U$  such that  $(h^{-1})^*u$  is absolutely continuous (uniformly  $(n-1)$ -locally integrable, respectively) on  $h(U \cap \Omega)$ .*

Let  $u$  and  $f$  be locally integrable forms on  $M$ , having degrees  $(n-1)$  and  $n$ , respectively. Recall that  $du = f$  on a open set  $\Omega$  of  $M$  in the distribution sense, if for any  $\varphi \in C_0^1(\Omega)$ ,

$$\int_M d\varphi \wedge u = - \int_M \varphi f.$$

**THEOREM 5.3.** *Let  $\Omega$  be a  $C^1$  domain of  $M$ , and  $u$  a  $(n-1)$ -form compactly supported in  $M$ . Assume that  $u$  is uniformly  $(n-1)$ -locally integrable and absolutely continuous on  $\Omega$ , and that the singular set*

$$\mathcal{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \text{supp } u$$

*has  $(n-1)$ -dimensional Hausdorff measure zero.*

*If  $u$  is integrable on  $\partial\Omega$  and  $du$  (taken in the sense of distribution theory) is integrable on  $\Omega$ , then*

$$\int_{\partial\Omega} u = \iint_{\Omega} du.$$

*Proof.* Using a smooth partition of unity and then working in local coordinates we can assume that  $M = \mathbf{R}^n$ . In this case, the conclusion is provided by Theorem 4.1.  $\square$

Note that, here again it suffices to have the “uniform” part of the local  $(n-1)$ -integrability condition for  $u$  fulfilled only on a small neighborhood of  $\mathcal{S}(u)$  (cf. also Remark 4.3).

**DEFINITION 5.4.** *A closed subset  $A$  of  $M$  is said to have an almost regular boundary if  $A$  coincides with the closure of its interior and if there exist a family  $(S_i)_{i \in I}$  of  $C^1$  submanifolds of  $M$  and a locally finite family  $(C_i)_{i \in I}$  of compact subsets of  $M$  such that:*

(1)  $C_i \subset S_i$ , for any  $i \in I$ , and  $\mathring{C}_i \cap \mathring{C}_j = \emptyset$ , for all  $i \neq j$  (the interiors are taken in  $S_i$  and in  $S_j$ , respectively);

(2)  $C_i \cap C_j$  has  $(n-1)$ -dimensional Hausdorff measure zero for all  $i \neq j$ ;

(3)  $\partial A = \cup_{i \in I} C_i$ .

Note that if  $A$  has an almost regular boundary, then

$$\Omega := \mathring{A} \cup \left( \bigcup_{i \in I} \mathring{C}_i \right)$$

(the interior of  $A$  is taken in  $M$ ) is a  $C^1$  domain with border  $b\Omega = \cup_{i \in I} \mathring{C}_i$ . If  $u$  is an integrally continuous  $(n-1)$ -form on  $M$ , it follows that  $u$  is integrable on each oriented submanifold  $\mathring{C}_i$  (with the standard orientation induced by  $\mathring{A}$ ). Since  $\partial C_i$  has zero measure in  $S_i$ , we can define

$$\int_{\partial A} u := \sum_{i \in I} \int_{\mathring{C}_i} u$$

whenever  $A \cap \text{supp } u$  is compact. Hence, without further proof, we can state the following.

**THEOREM 5.5.** *Let  $A$  be a subset of  $M$  with an almost regular boundary. If  $u$  is a  $(n-1)$ -form which is uniformly  $(n-1)$ -locally integrable on  $M$ , absolutely continuous on  $M$ , and for which  $A \cap \text{supp } u$  is compact, then  $u$  is integrable on  $\partial A$ ,  $du$  is integrable on  $\mathring{A}$  and*

$$\int_{\partial A} u = \iint_{\mathring{A}} du.$$