1. Introduction

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ON THE COHOMOLOGY OF COMPACT LIE GROUPS

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ABSTRACT. We give a new computation of the cohomology of a Lie group that some mathematicians may find to be shorter and more elementary than previous approaches. The main new ingredient is a result of L. Solomon on differential forms invariant under a finite reflection group. The cohomology is shown to have a bi-grading which has several interpretations.

1. INTRODUCTION

Let G be a compact connected Lie group, and let T be a maximal torus in G. We denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{t}$. Let W be the Weyl group of T in G. Then W acts on $\mathfrak{t}$ as a group generated by reflections about the kernels of the roots of $\mathfrak{t}$ in $\mathfrak{g} \otimes \mathbf{C}$. It has been known since the first half of this century that the cohomology ring $H(G)$, with real coefficients, is an exterior algebra with generators in degrees $2m_1 + 1, \ldots, 2m_l + 1$, where $m_1 + 1, \ldots, m_l + 1$ are the degrees of homogeneous generators of the ring of W-invariant polynomial functions on $\mathfrak{t}$. In particular, the Poincaré polynomial of G is $(1 + t^{2m_1+1}) \cdots (1 + t^{2m_l+1})$, and G has the cohomology of a product of odd-dimensional spheres.

Despite its age and familiarity, it is not easy to find a proof of this theorem in the literature. There are many beginnings and sketches in the textbooks, but the difficult part, namely the remarkable connection between degrees of invariant polynomials and Betti numbers, usually goes unproven. One reason is that the standard proofs (for example, [Bo2], [Ch], [L]) require substantial algebraic preliminaries on Hopf algebras, spectral sequences, and differential algebras. (See [Bo1] and [Sam] for historical surveys.)

We offer here a new but less sophisticated computation of the cohomology of a Lie group, avoiding the above algebraic techniques. Instead we use

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standard Lie theory and more invariant theory than is customary. We have tried to give a fairly complete treatment, in which it is seen that only a few ideas are used repeatedly. The most serious omission of proof is that of Chevalley's theorem on invariant polynomials, but that has many accessible references. Hopefully, enough background has been included to make the whole story coherent to someone with a basic knowledge of Lie groups and differential topology.

A sketch of our computation of $H(G)$ goes as follows. Consider the manifold M consisting of pairs (g, T'), where $g \in G$ and T' is a maximal torus in G which contains g. There is a natural map $\psi: M \to G$, known already to Weyl, given by conjugation. We make the apparently new observation that ψ induces an isomorphism of real cohomology rings $H(M) \simeq H(G)$. This uses only standard facts gleaned from the differential of ψ. One could also invoke the spectral sequence of the fibration $G \to G/T$. This spectral sequence was shown by Leray to degenerate at E^3. It in fact has a spectral subsequence (the W-invariants) which already degenerates at E^2 and still computes $H(G)$ (see (6.4) below).

We still have to compute the cohomology of M. It is easy to see that $H(M) = [H(G/T) \otimes H(T)]^W$. The ring $H(T)$ is naturally isomorphic to the exterior algebra of $\mathfrak{t}^*$, and $H(G/T)$ is isomorphic to the space $\mathscr{H}$ of W-harmonic polynomials on $\mathfrak{t}$, according to a famous theorem of Borel. For completeness, we give a proof of this in the same elementary, if less efficient spirit. As with all proofs, the essential thing is to show that the odd cohomology of G/T vanishes. We do this with a direct generalization of the Morse-theoretic computation of the cohomology of the two-sphere.

So now we are down to invariant theory, and must compute $[\mathscr{H} \otimes \Lambda \mathfrak{t}^*]^W$. This follows immediately from Solomon's determination of the W-invariant differential forms on $\mathfrak{t}$ with polynomial coefficients, which in turn depends on Chevalley's well-known description of W-invariant polynomials. This gives us the desired connection between degrees of W-invariants and Betti numbers of G. Solomon's result also leads to pretty formulas for the multiplicities of the W-modules $\Lambda^q \mathfrak{t}^*$ in spaces of harmonic polynomials (see (3.8)), as well as a generalization of a classical result on the Jacobian of the basic invariants (see (3.9)).

The paper is organized as follows: First the structure of G and its adjoint representation is recalled, then comes invariant theory, followed by the proof of Borel's theorem, finishing with the computation of $H(G)$ and some remarks on its natural bigrading. Throughout, cohomology has real coefficients.