8. Congruence subgroups and normal subgroups

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THEOREM 5. *Suppose that* $A = D$ *is not a totally definite quaternion algebra. Then there are isomorphisms*

$$H^k(\Gamma, M) \cong H^{k+g}(\Gamma, M)$$

for all Γ*-modules and* M *and* $k > r(D)$. *Moreover,* $r(D) + 1$ *is the smallest dimension from which on* $H^*(\Gamma, -)$ *is n-periodic.*

The last statement follows from the facts that (1) Γ is a virtual duality group and hence $H^{r(D)}(\Gamma, \mathbf{Z}\Gamma)$ is torsion free ([Br], VIII 11.2) whereas (2) for any group Γ with $vcd\,\Gamma = k$, $H^m(\Gamma)$ is torsion for $m > k$ ([Se 3], p. 101).

It should be possible to refine the period g in the theorem by one more directly derived from the periods of the finite subgroups of Γ as in the number field case where the period equals 2 if there are nontrivial torsion units of norm 1 (in which case n must be even!).

REMARK. We have seen (in the general case, Γ torsionfree) that

$$H^*(\Gamma, -) = H^*(X(\Gamma), -)\ .$$

Taking real coefficients (with trivial Γ-action) the latter groups are, by de Rham's theorem, given by differential forms on $X(\Gamma)$; these in turn correspond to Γ-automorphic forms on X. In this way, the real cohomology of Γ becomes part of the theory of automorphic forms.

8. CONGRUENCE SUBGROUPS AND NORMAL SUBGROUPS

Recall that we have defined

$$\Gamma(m) = \text{kernel of } (\Gamma \to (\Lambda/m\Lambda)^{\times})\ ,$$

the congruence subgroup of level m of Γ. Obviously $\Gamma(m)$ has finite index in Γ. The following question is classical: does every subgroup of finite index of Γ contain a congruence group?

Let us say that Γ satisfies (CP) if this is so. Let $\Lambda \subset \Lambda'$. If Γ' satisfies (CP), so does Γ. To prove the converse, it suffices to show that every $\Gamma(n)$ contains a $\Gamma'(m)$. This will be so if Γ contains a $\Gamma'(m)$. But there is $m \in \mathbf{N}$ with $m\Lambda' \subset \Lambda$, and this implies $\Gamma'(m) \subset \Lambda \cap \Gamma' = \Gamma$. Thus, property (CP) depends only on A.

For $A = K$ a number field, (CP) has essentially been proved by Chevalley [Ch]. Let $H < R^{\times}$ be of finite index, and $H_0 < R^{\times}$ any congruence subgroup. Then $H_0^k \subset H$ for some $k \in \mathbf{N}$; so it suffices to show that any power of a congruence group contains a congruence group. This follows at once from ([Ch], Th. 1). By an argument presented earlier, this allows us to reduce the problem to $S\Gamma$. If $D = K$, the answer is given by results of Bass-Milnor-Serre [BMS] and Serre [Se 5]. (Of course, one defines congruence groups with respect to ideals of R. For (CP) to hold, this makes no difference since every ideal of R contains an nR, $n \in \mathbf{Z}$.)

THEOREM 6. *Assume* $D = K$.

(a) [BMS] *If* $n \geqslant 3$, $SL_n(R)$ *has property (CP) if and only if* K *has a real embedding.*

(b) [Se 5] $SL_2(R)$ *has property (CP) if and only if* $r(K) > 0$.

The only K with $r(K) = 0$ are $K = \mathbf{Q}$ and $K =$ imaginary quadratic. The failure of (CP) for $SL_2(\mathbf{Z})$ is classic (see [Ne], p. 149). A discussion of the Bianchi case is given in [F, p. 200 ff.].

The case where $K \neq D$ and $n > 1$ was treated by Bak and Rehmann [BR]. They give an S-arithmetical result, from which we extract:

THEOREM 7. *Suppose* $K \neq D$ *and* $n \geqslant 2$*; if* $n = 2$ *suppose* $K \neq \mathbf{Q}$ *and that* D *is not a definite quaternion algebra. Then (CP) holds.*

The remaining cases seem still to be unsettled. Note that in the definite quaternion case the problem becomes trivial since $S\Gamma$ is finite.

The congruence groups are special since normal subgroups having finite index. It also makes good sense to ask whether every noncentral normal subgroup of $S\Gamma$ has finite index. This too may be asked more generally for discrete subgroups of Lie groups, and definite results have been obtained by Margulis. We specialize these to the case in question. As above, we can give no details of the proofs, which are extremely involved, and refer the reader to Margulis' monumental volume [M]. We make the following assumptions, excluding $A = K$:

$$(H)\quad \begin{cases} \text{(i) if } n = 1, \text{ then } D \text{ splits at all infinite primes of } K; \\ \text{(ii) if } K = \mathbf{Q}, \text{ then } k \geqslant 3, \text{ and } A \neq M_2(D),\ D \text{ definite quaternion.} \end{cases}$$

This means that the norm-1-group SG has no anisotropic factor and has rank $\geqslant 2$. It is clear that — in the terminology of [M] — $S\Gamma$ is an irreducible lattice of SG. Applying Theorems (2) and (4) from the introduction of [M], we obtain

THEOREM 8. *Assume* (H). *Then every noncentral normal subgroup of* $S\Gamma$ *has finite index.*

COROLLARY. $S\Gamma/[S\Gamma, S\Gamma]$ *is finite.*

Again, it is classic that the theorem fails for $SL_2(\mathbf{Z})$; for instance, if $n \gg 0$, the verbal subgroup generated by all $6n$-th powers has infinite index ([Ne], p. 143). The corollary holds for $SL_2(\mathbf{Z})$ but fails for torsion free subgroups (which are free). Theorem 7 however carries over to subgroups of finite index.

Two more topics from the general theory of arithmetic groups, which could be specialized to unit groups, are subgroup rigidity and strong approximation. But having promised to keep as near as possible to the unit groups "themselves", we omit this.

9. THE BASS UNIT THEOREM

In his paper [Ba 2] Bass has proved (among other things) a far reaching generalization of Dirichlet's unit theorem which — together with the results of sections 3 and 7 — is certainly one of the strongest general results we have about Γ. The core of the proof is a deep stability theorem from K-theory; we will indicate how it implies the theorem but will say little about its proof. We begin with the relevant definitions. For any ring A, define

$$K_1(A) = \varinjlim GL_n(A)/[GL_n(A), GL_n(A)] ,$$

where the direct limit is taken with respect to the embeddings

$$GL_n(A) \to GL_{n+1}(A), \quad x \to \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} .$$

One may also write

$$K_1(A) = \varinjlim \quad GL_n(A)/\tilde{E}_n(A) ,$$

where $\tilde{E}_n(A)$ is the normal subgroup generated by the elementary matrices; this is Whitehead's lemma. Further, with $K_0(A)$ denoting the Grothendieck group of finitely generated projective A-modules, we put

$$R_0(A) = \mathbf{R} \otimes K_0(A), \; R_1(A) = \mathbf{R} \otimes K_1(A) .$$

Now we turn to algebras and allow A to be semisimple. Let $\Lambda \subset A$ be an order. Any $A_{\mathbf{R}}$-module V (of finite dimension) gives rise to a homomorphism

$$\Lambda \to \mathrm{End}_{\mathbf{R}} V, \text{ hence by functoriality } K_1(\Lambda) \to K_1(\mathrm{End}_{\mathbf{R}} V) .$$