2. Elementary Properties

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.04.2024**

2. Elementary Properties

(1) Let N denote the regular norm $A^\times \to \mathbf{Q}^\times$. It is easy to see that $\Gamma = \{x \in \Lambda \mid N^2(x) = 1\}$.

If we specify a $\mathbf{Z}$-basis of Λ, this becomes a polynomial equation in the coefficients of x with respect to this basis, and the elements of Γ correspond precisely to the integral solutions. This shows that Γ is an *arithmetic group*, and thereby makes available all the general results on this class of groups. (A reference ideally suited to the present theme is Serre's survey article [Se4]; we also mention [Pl].) In fact, a good deal of the present paper will be concerned with specifying the general results to the case of unit groups.

(2) *Let* $\Lambda \subset \Lambda'$ *be orders in* A *with unit groups* Γ, Γ'. *Then* $\Gamma = \Gamma' \cap \Lambda$, *and* $|\Gamma' : \Gamma|$ *is finite.*

Proof. For any $x \in \Gamma$ we have $x^{-1} \in \mathbf{Z}[x]$ since x is a zero of a monic integral polynomial with constant term ± 1. This proves the first statement. For the second, assume that, for $x, y \in \Gamma'$, we have

$$x - y = mz, \quad \text{where} \quad m = |\Lambda' : \Lambda|, z \in \Lambda' .$$

Then

$$xy^{-1} = 1 + mzy^{-1} \in \Gamma' \cap \Lambda = \Gamma .$$

This shows that

$$|\Gamma' : \Gamma| \leqslant |\Lambda' : m\Lambda| = (\dim A)^m .$$

(2) allows us to reduce all questions concerning virtual properties of Γ to arbitrary orders in simple algebras. (A group is said to have a property virtually if a subgroup of finite index has that property). Finite presentability is such a property: if $\Gamma_0 \subset \Gamma$, $|\Gamma : \Gamma_0|$ finite, has a finite presentation, then so has, by Reidemeister-Schreier, the intersection of its conjugates, which is normal; now use the fact that the class of finitely presented groups is closed under extensions ([J], p. 187, Th. 1).

(3) Γ *is virtually torsion free.*

Proof. It is easy to see that there is an upper bound, and consequently a lowest common multiple N for the orders of torsion elements $x \in A^\times$; all such $x \neq 1$ satisfy

$$x^{N-1} + x^{N-2} + \dots + x + 1 = 0 .$$

For $n \in \mathbf{N}$ let

$$\Gamma(n) = \text{kernel of } (\Gamma \to (\Lambda/n\Lambda)^{\times})$$

the congruence group mod n; this is a normal subgroup of finite index. Obviously $\Gamma(n)$ is torsion free for $n > N$. With more effort, one can do much better: the regular representation injects $\Gamma(n)$ into the congruence group mod n in $GL_m(\mathbf{Z})$, $m = \dim A$, and Minkowski has shown that this is torsion free for $n > 2$ [Mi].

(4) Γ *contains only finitely many isomorphism classes of finite subgroups.*

Proof. If $\Gamma_0 < \Gamma$ is torsion free and normal of finite index, then every finite subgroup of Γ is isomorphic to a subgroup of Γ/Γ_0.

Later, we will show more: Γ contains only finitely many conjugacy classes of finite subgroups.

(5) Γ *is residually finite, that is, for every* $x \in \Gamma, x \neq 1$, *there is a normal subgroup* Γ_0 *of finite index such that* $x \notin \Gamma_0$.

Of course, almost all $\Gamma(n)$ will do. It follows that Γ is hopfian, that is, not isomorphic to a proper factor group (see [MKS], p. 116).

(6) Finally, let us mention here the following result due to Zassenhaus [Z2] (although it is not entirely elementary): Γ contains a solvable subgroup of finite index if and only if the Wedderburn components of A are number fields or definite quaternions over $\mathbf{Q}$.

Sketch of proof: the problem is readily reduced to simple A. The "If" part is then trivial.

Conversely, if matrices are involved, one knows that Γ has infinitely many subfactor groups of the form $SL_n(F)$, where F is a finite field. The same is therefore true of any subgroup of finite index. In the skew field case, the argument is more intricate; we refer to [Z2].

3. Finite generation: classical reduction theory

The most basic fact about Γ is that it is finitely generated; this is even valid for arbitrary arithmetic groups, as has been proved by A. Borel and Harish-Chandra in the fundamental paper [BHC]. Here I shall describe the classical approach, carried out by Siegel [S1], who completed earlier work of