

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 40 (1994)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AN EXPOSITION OF POINCARÉ'S POLYHEDRON THEOREM
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Anhang: 10. Appendix
DOI: <https://doi.org/10.5169/seals-61108>

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10. APPENDIX

This appendix contains results due to Brian Bowditch, published here with his permission.

We recall that a finite-sided closed convex cell of \mathbf{H}^{n+1} is said to be *pyramidal* at an ideal point p if any two faces whose closures contain p meet in \mathbf{H}^{n+1} . The intersection of such a convex cell with a horosphere centred at p is a euclidean finite-sided closed convex cell of dimension n (provided the horosphere only meets faces which have p as an ideal point). One way to see this is to use the upper half-space model with p equal to the point at infinity. Conversely, given a convex finite-sided n -dimensional euclidean cell, we can think of this cell as lying in a horosphere which is a horizontal subspace in the upper half-space model. This gives rise to an $(n+1)$ -dimensional hyperbolic convex cell, by taking the intersection of vertical half-spaces determined by the half-spaces defining the euclidean convex cell. We use the names “pyramidal” and “non-pyramidal” for convex euclidean cells if the corresponding hyperbolic cells are pyramidal or non-pyramidal respectively. A euclidean convex cell is non-pyramidal if and only if it has disjoint faces. If a euclidean cell is pyramidal, then there is a face which is the intersection of all other faces, that is there is a unique minimal face. A pyramidal euclidean n -cell is the product of an i -dimensional cell with the cone on a spherical $(n-i-1)$ -dimensional cell. (The cone point is placed at the centre of the $(n-i-1)$ -dimensional sphere.)

Let M be a connected euclidean similarity n -dimensional manifold which is the union of a locally finite set of closed subsets $\{X_i\}$. Each X_i has an induced similarity structure which is isomorphic to that of a closed finite-sided euclidean convex polyhedron. There are only a finite number of distinct similarity classes of X_i . The intersection of any face of any X_i with any face of any X_j is a common face of each. This implies that M has the structure of a locally finite polyhedral cell complex. Let G be a group of similarities of M which preserve the cell structure. Suppose that the number of orbits of non-pyramidal polyhedra is finite.

THEOREM 10.1 (Bowditch). *Under the above assumptions, the number of orbits of cells is finite. Moreover, the number of orbits is bounded in terms of the number of orbits of non-pyramidal cells and the geometry (up to similarity) of the X_i .*

Bowditch has suggested that if there is one or more pyramidal polyhedral cell, then one should be able to prove that G is a finite group. It would follow

that G consists of euclidean isometries and that M contains only a finite number of cells. This conjecture remains open.

Proof of 10.1. Let X be the union of the non-pyramidal cells in M , and let Y be the union of cells which meet X . Note that $X \subset Y$.

Now suppose there is a top-dimensional cell which is not in Y and let σ be its unique minimal face. Then σ is similar to \mathbf{R}^i for some i . If α is any cell meeting σ , then $\sigma \subset \alpha$ since σ is minimal. Clearly α is not in X . Therefore σ is the unique minimal face of α . We have seen above that α is the product of σ and the cone on a convex subset \mathbf{S}^{n-i-1} . It follows that the union of the cells meeting σ is the product of σ with the cone on \mathbf{S}^{n-i-1} . It follows that the cell structure of M is finite, G is a finite group and $X = \emptyset$. The other possibility is that $Y = M$.

Let $K \subset X$ be a finite union of cells such that $GK = X$. The cell structure of M is locally finite, with a bound for the number of cells in any small neighbourhood being given by the geometry of the X_i . The number of cells of M which meet K is bounded by the number of cells of K and the maximum possible number of cells meeting a fixed small neighbourhood of any fixed point of K . This gives an upper bound for the number of orbits of cells of M under the action of G in case $Y = M$. If $X = Y = \emptyset$, then the number of cells of M is bounded by the geometry of the X_i . \square

We apply Theorem 10.1 to find out a little more about the spaces that arise in Poincaré's Theorem. Suppose the hypotheses $\text{Pairing}(\mathcal{P}, R, A)$, $\text{Connected}(\mathcal{P}, R)$, $\text{Finite}(\mathcal{P})$ and $\text{Cyclic}(\mathcal{P}, R, A)$ are satisfied for a set of convex cells (see Definition 2.8) in \mathbf{H}^n . To each convex cell we adjoin the ideal points, so as to obtain a compact space. The face-pairings are defined on the closures of the faces. Let \bar{Q} be the quotient of the disjoint union of the extended cells by the face-pairings, endowed with the quotient topology.

THEOREM 10.2. \bar{Q} is a compact hausdorff space.

Proof of 10.2. Let X be the disjoint union of the closures of the convex cells. So X is compact and hausdorff. We first show that the inverse image of a point under the quotient map $X \rightarrow \bar{Q}$ is a finite set. This is clear from Theorem 4.13 for any point which is not an ideal point. For an ideal point p , we can construct a similarity manifold to which Theorem 10.1 applies, by developing a horosphere centred at p into \mathbf{R}^{n-1} . More details, which will help the interested reader with the construction of the similarity manifold, are given in the discussion of Definition 6.2.

A pyramidal cell in \mathbf{R}^{n-1} corresponds to a convex cell in \mathbf{H}^n together with an ideal point p in its boundary, such that any two faces with closures containing p meet inside \mathbf{H}^n . A non-pyramidal cell corresponds to a convex cell in \mathbf{H}^n and an ideal point p contained in the closures of two non-intersecting faces of the convex cell. The hypothesis needed in order to apply Theorem 10.1, that there are only a finite number of orbits of non-pyramidal cells, comes from the fact that there are only a finite number of pairs of faces and therefore only a finite number of pairs of non-intersecting faces which meet at infinity.

It follows that the inverse image in X of any point of \bar{Q} is finite. Moreover the number of points in the inverse image is bounded by a fixed integer N . Two points $x, y \in X$ are mapped to the same point of \bar{Q} if and only if there is a sequence (x_0, \dots, x_n) such that $x = x_0$, $y = x_n$ and $x_{i+1} = A(F_i)(x_i)$, where $x_i \in F_i$ and $x_{i+1} \in R(F_i)$. (Here (R, A) is the glueing data.) We may take $n \leq N$. It follows easily from compactness and the finiteness of the situation that the map $X \rightarrow \bar{Q}$ is closed. Therefore \bar{Q} is hausdorff. \square

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