

Appendix: Recurrent points

Objektyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

V_1 -invariant subsets of X and $v_2(t_k)Y_k \subseteq X$ for all k . Also it is easy to see that $\{t_k | k \geq 1\}$ contains either all positive rational or all negative rational numbers. Now let $Y' = \bigcap_{k=1}^{\infty} Y_k$. Since $\{Y_k\}$ is a decreasing sequence of compact subsets, Y' is nonempty. Now if $\{t_k | k \geq 1\}$ contains all positive rational numbers then $v_2(r)Y' \subseteq X$ for all positive rational numbers r and hence by continuity $V_2^+ Y' \subseteq X$ and, similarly, in the alternative case $V_2^- Y' \subseteq X$. This completes the proof of the theorem.

APPENDIX: RECURRENT POINTS

For a compact metric space X we denote by $C(X)$ the space of all continuous real-valued functions on X equipped with the sup-norm topology and by $C(X)^+$ the subset of $C(X)$ consisting of all nonnegative functions; the supremum norm of $f \in C(X)$, namely $\sup\{|f(x)| | x \in X\}$, will be denoted by $\|f\|$. By an integral on $C(X)$ we mean a linear functional on $C(X)$ which takes nonnegative values on $C(X)^+$. For an integral Λ on $C(X)$ the *support* of Λ is defined to be the subset of X consisting of all $x \in X$ such that $\Lambda(f) > 0$ for any $f \in C(X)^+$ for which $f(x) > 0$; the support is easily seen to be a closed subset of X . It can also be verified by a simple point-set topological argument that if Λ is an integral on $C(X)$ and $f \in C(X)$ vanishes on the support of Λ then $\Lambda(f) = 0$. If Λ is an integral on $C(X)$, where X is a compact metrizable space, and X' is the support of Λ then there exists a unique integral Λ' on $C(X')$ such that $\Lambda'(f|_{X'}) = \Lambda(f)$ for all $f \in C(X)$, where $f|_{X'}$ denotes the restriction of f to X' ; this follows from the Tietze-Urysohn extension theorem (cf. [D], (4.5.1)) and the above mentioned property of the support. We note also that the support of Λ' as above is the whole of X' .

For any homeomorphism ϕ of a compact (metrizable) space X an integral Λ on $C(X)$ is said to be ϕ -invariant if $\Lambda(f \circ \phi) = \Lambda(f)$ for all $f \in C(X)$; clearly the support of a ϕ -invariant integral on $C(X)$ is a ϕ -invariant (closed) subset of X .

Proof of Proposition 1.7. We fix a dense sequence in $C(X)$, say $f_j, j = 1, 2, \dots$. Let $x_0 \in X$. Given f_j , for any sequence $\{m_k\}$ of natural numbers $m_k^{-1} \sum_{i=0}^{m_k-1} f_j \circ \phi^i(x_0)$ is a bounded sequence and therefore admits a convergent subsequence. Using a standard procedure (finding $\{m_k^{(j)}\}$, with each sequence a subsequence of the previous one, such that the corresponding sequence for f_j as above converges and considering $\{m_k^{(k)}\}$) we get a sequence $\{n_k\}$ of natural numbers such that $n_k^{-1} \sum_{i=0}^{n_k-1} f_j \circ \phi^i(x_0)$ converges for all j ; also, the limit is between $-\|f_j\|$ and $\|f_j\|$. Since $\{f_j\}$ is dense

in $C(X)$ this readily implies that $n_k^{-1} \sum_{i=0}^{n_k-1} f \circ \varphi^i(x_0)$ converges for all $f \in C(X)$; let c_f be the limit corresponding to f . Then it can be verified that $\Lambda: C(X) \rightarrow \mathbf{R}$ defined by $\Lambda(f) = c_f$, for all $f \in C(X)$, is a φ -invariant integral on $C(X)$. Also clearly Λ is not identically zero and therefore by our observations above, the support, say X' , is a nonempty closed φ -invariant subset of X and further $C(X')$ admits an integral with full support (namely X') which is invariant under the restriction of φ to X' . Replacing X as in the hypothesis by X' we may without loss of generality assume that $C(X)$ admits a φ -invariant integral whose support is X ; in the rest of the argument we let Λ be any such integral.

Now suppose that there do not exist any recurrent points for φ . Let $\rho(\cdot, \cdot)$ be the metric on X . Let θ be the function on X defined by $\theta(x) = \inf\{\rho(\varphi^i(x), x) \mid i = 1, 2, \dots\}$, for all $x \in X$. There being no recurrent points means that $\theta(x) > 0$ for all $x \in X$. For each natural number k let $E_k = \{x \in X \mid \theta(x) \geq 1/k\}$. Then each E_k is a closed subset of X and $X = \cup E_k$. Therefore by the Baire category theorem there exists a k such that E_k has an interior point in X . In particular, there exists an open ball, say A , of radius at most $1/3k$ contained in E_k . The definition of E_k and the condition on the radius of A then imply that the sets $\varphi^i(A)$, $i \in \mathbf{Z}$, are mutually disjoint. Now let $x \in A$ and let $f \in C(X)^+$ be such that $f(x) > 0$ and the support of f (the closure of the set $\{y \in X \mid f(y) > 0\}$) is contained in A . For each natural number n let $S_n(f) = \sum_{i=0}^{n-1} f \circ \varphi^i \in C(X)$. The disjointness of $\varphi^i(A)$, $i \in \mathbf{Z}$, implies that, for any n , $\|S_n(f)\| = \|f\|$. Also, by the φ -invariance of Λ we have $\Lambda(S_n(f)) = n\Lambda(f)$. Hence $\Lambda(f) = \Lambda(S_n(f))/n \leq \|S_n(f)\| \Lambda(1_X)/n = \|f\| \Lambda(1_X)/n$ for all n , where 1_X denotes the constant function with value 1. But this implies that $\Lambda(f) = 0$ contradicting the assumption that the support of Λ is the whole of X . This proves the proposition.

Acknowledgement: Thanks are due to J. Aaronson and M.G. Nadkarni for a discussion on recurrent points and to Gopal Prasad and Nimish Shah for useful comments on an earlier version.

REFERENCES

- [DM-1] DANI, S.G. and G.A. MARGULIS. Values of quadratic forms at primitive integral points. *Invent. Math.* 98 (1989), 405-424.
- [DM-2] DANI, S.G. and G.A. MARGULIS. Values of quadratic forms at integral points; an elementary approach. *L'Enseignement Math.* 36 (1990), 143-174.