

## 2. The abstract algebras

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## 2. THE ABSTRACT ALGEBRAS

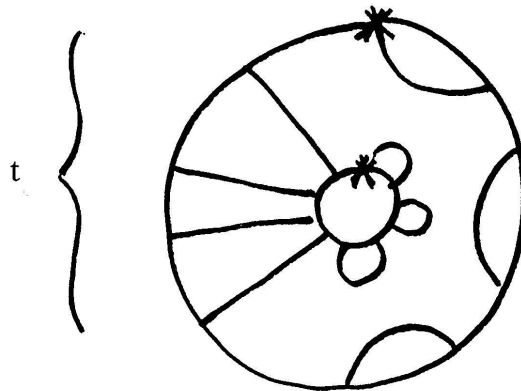
The (abstract) Brauer algebra with parameter  $\delta \in \mathbf{C}$ ,  $B(n, \delta)$ , is the algebra with basis the set of all  $(n, n)$ -diagrams and multiplication law  $\alpha\beta = \delta^{n(\alpha, \beta)} \alpha \circ \beta$ . We could say it is the twisted monoid group algebra for the monoid  $(D(n, n), \circ, 1)$  and the cocycle  $\delta^n$ . We have thus at our disposition two other series of abstract algebras with parameter, subalgebras of the Brauer algebra:

$P(n, \delta) =$  The subalgebra spanned by planar diagrams  
also called the Temperley-Lieb algebra  $TL(n, \delta)$ ,  
in fact invented as diagrams by Kauffman ([K]).

$A(n, \delta) =$  The subalgebra spanned by annular diagrams.

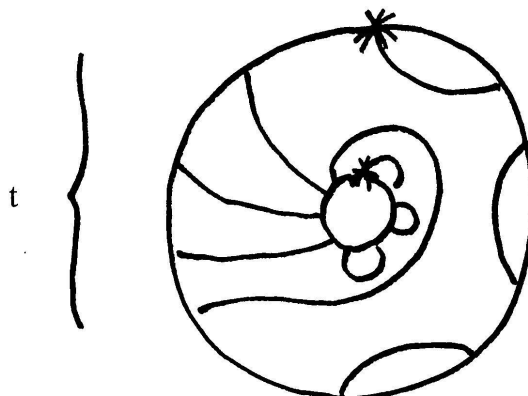
The structure of the Brauer algebra has been studied extensively. See [W], [HW] for much information, and  $P(n, \delta)$  is particularly well understood (see [GW], [GHJ]). In this section we will give the structure of  $A(n, \delta)$  whenever it is semisimple (over  $\mathbf{C}$ ). It will be worthwhile to call the algebra simply  $A(n)$  in this section since we will only consider a fixed  $\delta (\neq 0)$ .

*Definition 2.1.* (i) We call  $E(n, t)$  the diagram (in  $\mathcal{A}(n, n; t)$ )



(so that  $E(n, n) = 1$ ).

(ii) We call  $V(n, t)$  the diagram (in  $\mathcal{A}(n, n; t)$ )



(so that  $u = V(n, n)$  and  $E(n, 0) = V(n, 0)$ ).

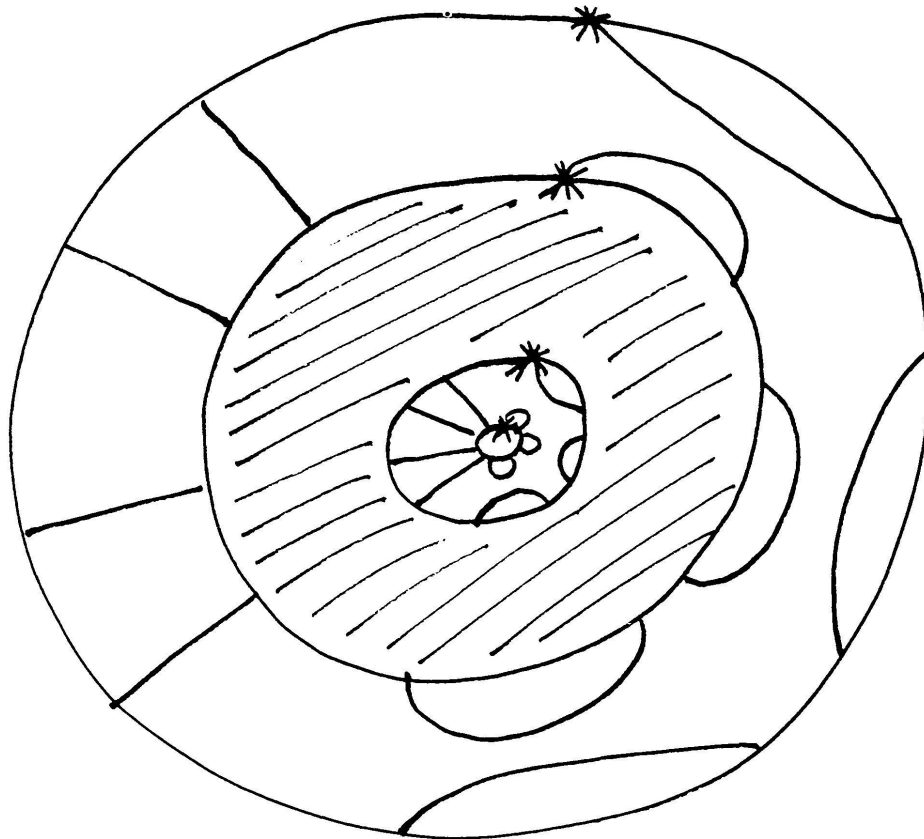
Note: the role of  $*$  is unimportant, it serves only to have a well defined element.

LEMMA 2.2. Let  $e_t \in A(n)$  be  $\delta^{-\binom{n-1}{2}} E(n, t)$  and  $v_t \in A(n)$  be  $\delta^{-\binom{n-t}{2}} V(n, t)$ . Then

- (i)  $e_t^2 = e_t$ .
- (ii)  $(v_t)' = e_t$  (so  $e_t v_t = v_t e_t$ ).
- (iii)  $E(n, t) \circ \mathcal{A}(n, n) \circ E(n, t) \subset \cup_{j < t} \mathcal{A}(n, n; j) \cup \{V(n, n)^k \mid k = 0, 1, 2, \dots, t-1\}$ .
- (iv) If  $D \in \mathcal{A}(n, n; t)$ , there are  $D_1$  and  $D_2$  in  $\mathcal{A}(n, n, t)$  with  $D = D_1 \circ E(n, t) \circ D_2$ .

*Proof.* (i) and (ii) are evident from diagrams and the multiplication structure in  $A(n)$ .

(iii) For any  $D$  in  $\mathcal{A}(n, n)$ ,  $x = E(n, t) \circ D \circ E(n, t)$  is as below.



where there is any annular diagram in the intermediate annulus (shaded). But we see that if  $x$  has  $t$  through-strings, the intermediate system must connect all of the outer through-strings to one of the inner ones. Once one connection is fixed, all the others must follow in cyclic order, so  $x$  is a power of  $V$  (with respect to  $\circ$ ).

(iv) As in the proof of Corollary 1.16, we may write  $D = E_1 \circ E_2$  with  $E_1 \in \mathcal{A}(n, t; t)$ ,  $E_2 \in \mathcal{A}(t, n; t)$ . But then pulling the strings around in the middle and introducing  $\frac{n-t}{2}$  isolated circles we see that  $D$  admits the desired decomposition.  $\square$

We proceed to determine the structure of  $A(n, \delta)$  when it is semisimple. Note first that the through-strings give a filtration of  $A(n)$  by ideals.

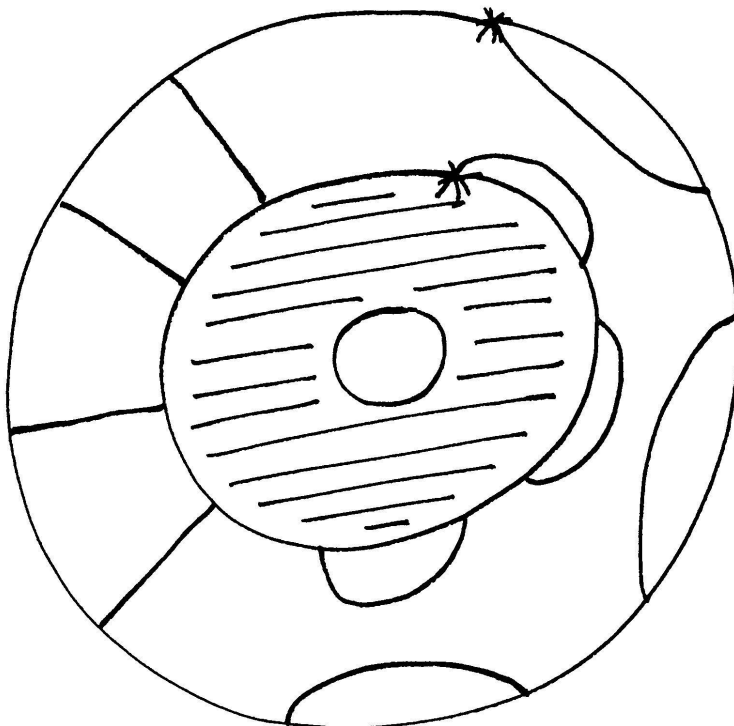
*Definition 2.3.*  $A(n; t)$  is the two-sided ideal linearly spanned by diagrams with  $\leq t$  through-strings.

Thus if  $A(n)$  is semisimple, it is isomorphic to the direct sum  $\bigoplus_{t=0}^n \frac{A(n; t)}{A(n; t-2)}$ , and to determine its structure it suffices to determine that of the quotients, which of course are all semisimple.

**THEOREM 2.4.** *If  $\delta$  is such that  $A(n, \delta)$  is semisimple,*

$$\frac{A(n, t)}{A(n, t-2)} \cong \begin{cases} \text{A matrix algebra of size } \text{cat} \left( \frac{n}{2} \right) & \text{if } t = 0 \text{ and } n \text{ even.} \\ \text{The sum of } t \text{ matrix algebras of size } \binom{n}{\frac{n-t}{2}} & \text{if } t > 0 \\ & \text{(and } n-t \text{ even).} \end{cases}$$

*Proof.* Suppose first  $t > 0$ . Let  $A$  stand for  $A(n, t)/A(n, t-2)$  for short and let it be isomorphic to  $\bigoplus_{i=1}^r M_{d_i}(\mathbf{C})$ . Identify elements of  $A(n, t)$  with their classes modulo  $A(n, t-2)$ . Then by (iv) of Lemma 2.2, the 2-sided ideal generated by  $e_t$  is all of  $\bigoplus_{i=1}^r M_{d_i}(\mathbf{C})$  so we can write  $e_t = \bigoplus_{i=1}^r p_i$  with  $p_i$  a non-zero idempotent in each  $M_{d_i}(\mathbf{C})$ . But  $A$  is linearly spanned by the diagrams in  $\mathcal{A}(n, n; t)$  so by (ii) and (iii) of 2.2,  $e_t A e_t$  is abelian of dimension  $t$ . Thus each of the  $p_i$ 's is a minimal idempotent,  $r = t$  and of course  $\sum_{i=1}^t d_i^2 = t \binom{n}{\frac{n-t}{2}}^2$  by (1.16). But also  $\mathcal{A}(n, n; t) \circ E(n, t)$  is exactly all diagrams of the form



so that the ones representing non-zero elements of  $A$  are in bijection with  $\mathcal{A}(n, t; t)$ . Hence  $\dim(Ae_t) = |\mathcal{A}(n, t; t)| = t \binom{n}{\frac{n-t}{2}}$ . However,  $(\bigoplus_{i=1}^t M_{d_i}(\mathbf{C})) (\bigoplus_{i=1}^t p_i)$  is a vector space of dimension  $\sum_{i=1}^t d_i$ , so we have

$$\sum_{i=1}^t d_i^2 = t \binom{n}{\frac{n-t}{2}} \quad \text{and} \quad \sum_{i=1}^t d_i = t \binom{n}{\frac{n-t}{2}}.$$

Thus each of the  $d_i$ 's is equal to  $\binom{n}{\frac{n-t}{2}}$  (e.g. by the ‘‘equality’’ case of the Cauchy Schwartz inequality  $(\sum d_i \cdot 1) \leq \sqrt{\sum d_i^2} \sqrt{t}$ ). This proves the theorem for  $t > 0$ . The case  $t = 0$  follows from the same argument, using  $\dim(\mathcal{A}(n, n; 0)) = \text{cat}(n)^2$  and  $\dim(\mathcal{A}(n, n; 0)e_0) = \text{cat}(n)$ .  $\square$

Note that one could avoid the slightly clumsy Cauchy-Schwartz argument by showing that the commutant of  $\mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$  is  $A(n)$ , which is not hard.

*Remark 2.5.* In fact it is clear from the proof that the algebra  $e_t(A(n, t)/A(n, t-1))e_t$  is naturally isomorphic to the group algebra  $\mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$ , so that the various matrix algebras in  $A(n, t)/A(n, t-2)$  are naturally indexed by the  $t$ -th roots of unity.

*Remark 2.6.* In view of 2.5, another way of stating Theorem 2.4 is to say that, if  $A(n, \delta)$  is semisimple, its irreducible representations are parametrised by

- (i) the number of through-strings  $t$
- (ii) a  $t$ -th root of unity  $\omega$ .

Moreover the irreducible representation  $\pi = \pi_{t, \omega}$  corresponding to  $(t, \omega)$  is characterised by the fact that  $\pi(v_t) = \omega\pi(e_t)$ , and may be given quite explicitly as follows:

If  $W$  is the vector space spanned by  $\mathcal{A}(t, n; t)$ ,  $W$  becomes an  $A(n) - \mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$  bimodule under the left and right action:

$$D \cdot E \cdot F = \begin{cases} D \circ E \circ F & \text{for } D \in \mathcal{A}(n, n) \text{ and } F \in \mathcal{A}(t, t; t), \text{ identified} \\ & \text{with } \mathbf{Z}/t\mathbf{Z}. \\ 0 & \text{if } D \circ E \text{ has } < t \text{ through-strings.} \end{cases}$$

Then if  $P_\omega = \frac{1}{t} \sum_{i=1}^t \omega^{-i} u^i$  ( $u$  as in 1.10),  $\pi_{t, \omega}$  is left multiplication on  $VP_\omega$ .

We give the structure of the subalgebra  $\overrightarrow{A(n)}$  of  $A(n)$  spanned by oriented diagrams. With obvious notation the result is

THEOREM 2.7. If  $\delta$  is such that  $\overrightarrow{A(n, \delta)}$  is semisimple, ( $n$  even),

$$\overrightarrow{A(n, t) / A(n, t-2)} \cong \begin{cases} \text{A matrix algebra of size } \text{cat} \binom{n}{2} & \text{if } t = 0. \\ \text{The sum of } \frac{t}{2} \text{ copies of a matrix algebra} \\ \text{of size } \binom{n}{\frac{n-t}{2}} & \text{if } t > 0. \end{cases}$$

*Proof.* One can simply repeat the proof of Theorem 2.4, the only difference being that the role of the element  $v$  would be played by  $v^2$ . One could also deduce 2.7 from 2.4 in several ways. One is to note that  $\overrightarrow{\mathcal{A}(n)}$  is the fixed point algebra for an involutive automorphism of  $\mathcal{A}(n)$  sending  $u$  to  $-u$ . Another way is to observe that the irreducible representations of  $\mathcal{A}(n)$  parametrised by  $(t, \omega)$  ( $t > 0$ ) remain inequivalent for  $\omega = \exp\left(\frac{2\pi\sqrt{-1}j}{n}\right)$ ,  $j = 0, 1, \dots, \frac{n}{2} - 1$  on restriction to  $\overrightarrow{\mathcal{A}(n)}$ . This is because  $v_t^2 = \omega^2 e_t$  in that representation. Then adding the sums of squares of the dimensions one gets the number of oriented diagrams by 1.20.  $\square$

Finally we make some remarks about generators and relations. As we saw in the introduction, if we put  $f_i = u^i e_{n-2} u^{-i}$  (and  $F_i = u^i E(n-2; n-2) u^{-i}$ ) for  $i = 1, 2, \dots, n$ , the  $f_i$ 's satisfy  $f_i^2 = f_i$ ,  $f_i f_{i\pm 1} f_i = \delta^{-\frac{1}{2}} f_i$  so that if  $g_i = q f_i - (1 - f_i)$  (for  $q + q^{-1} + 2 = \delta^2$ ), the map  $T_i \mapsto g_i$ ,  $\rho \mapsto u$  gives a homomorphism from the affine Hecke algebra of type  $A_n$  with parameter  $q$  onto the diagram algebra  $A(n, 2 + q + q^{-1})$ . Thus in particular we have constructed some very explicit irreducible representations of the affine Hecke algebra, for certain values of  $q$ .

One reason, besides subfactors, for looking at oriented diagrams in the even case is that they allow us to determine the subalgebra generated by  $f_1, f_2, \dots, f_n$  (or  $g_1, \dots, g_n$ ).

LEMMA 2.8. If  $n$  is even the following three algebras are equal (even if  $A(n, \delta)$  is not semisimple).

- (i) The subalgebra of  $A(n)$  generated by  $f_1, f_2, \dots, f_n$ .
- (ii) The two-sided ideal generated by  $f_1$  in  $A(n)$ .
- (iii)  $\overrightarrow{A(n, n-2)}$ .

*Proof.* The equality of (ii) and (iii) follows from a special (oriented) case of (iv) of 2.2.

The algebra of (iii) contains the  $f_i$ 's by definition. That (iii) implies (i) will follow if we can show that any element of  $\cup_{t < n} \mathcal{A}(n, n; t)$  is expressible as a product of  $F_i$ 's. That this is true for diagrams having a straight through-string is a well known fact about the Temperley-Lieb algebra. But if  $D$  is an oriented diagram with less than  $n$  through-strings, either  $D$  has zero through-string and we are in the Temperley-Lieb situation, or  $D \circ u^k$  has a straight through-string for some even  $k$ . Thus  $Du^k$  is a word on the  $F_i$ 's and it suffices to show that  $F_i u^2$  is a word on the  $F_i$ 's for all  $i$ . It follows from a picture that  $F_i u^{-2} = F_i F_{i+1} \dots F_n F_1 F_2 \dots F_{i-2}$ .  $\square$

*Remark 2.9.* We leave it to the reader to show that Lemma 2.8 is true without the  $\rightarrow$ 's if  $n$  is odd.

*Remark 2.10.* It follows from 2.8 that the elements  $v_t$  are in the algebra generated by the  $F_i$ 's for  $t < n$ . We record the expression

$$v_{n-2}^2 = F_n \circ F_1 \circ F_2 \circ \dots \circ F_n .$$

Thus rotations are unavoidable even if one is only interested in the structure of the algebra generated by the  $F_i$ 's.

### 3. THE BRAUER REPRESENTATION

So far we have begged the important question of when the algebra  $A(n, \delta)$  is semisimple. We do not have a complete answer for this but we shall show that it is semisimple whenever  $\delta$  is an integer  $\geq 3$ , (and that  $A(n, -2)$  is not semisimple for  $n \geq 3$ ) by using a representation onto a  $C^*$ -algebra which we will show to be faithful for such  $\delta$ . That the representation is faithful for  $n$  fixed and large integral (hence any large)  $\delta$  is rather easy.

*Definition 3.1.* Let  $V$  be a vector space of dimension  $k$  and basis  $w_1, w_2, \dots, w_k$ . If the diagram  $D \in D(n, n)$  has  $n$  connecting edges called  $\varepsilon$ , define  $\beta(D) \in \text{End}(\otimes^n V)$  by the matrix (with respect to the basis  $\{w_{a_1} \otimes w_{a_2} \otimes \dots \otimes w_{a_n} \mid a_i = 1, 2, \dots, k\}$  of  $\otimes^n V$ )

$$\beta(D)_{a_1 a_2 \dots a_n}^{a_{n+1} \dots a_{2n}} = \prod_{\varepsilon} \delta(a_{s(\varepsilon)}, a_{b(\varepsilon)})$$

where  $s(\varepsilon), b(\varepsilon)$  are the two ends of the edge  $\varepsilon$ , labelled from 1 to  $2n$ , and, just in this formula,  $\delta$  is the Kronecker  $\delta$ .

LEMMA 3.2.  $D \mapsto \beta(D)$  defines a homomorphism of  $B(n, k)$  (hence  $A(n, k)$ ) onto a  $C^*$ -subalgebra of  $\text{End}(\otimes^n V)$ .