0. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

L'Enseignement Mathématique, t. 40 (1994), p. 313-344

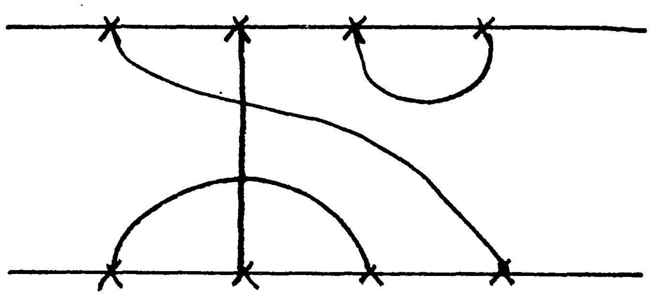
A QUOTIENT OF THE AFFINE HECKE ALGEBRA IN THE BRAUER ALGEBRA

by V.F.R. JONES[1])

ABSTRACT. The structure of a certain subalgebra of Brauer's centralizer algebra is given for all values of the parameter for which it is semisimple. The algebra admits a trace functional whose weights on the simple components of the algebra are calculated. The algebra may be exhibited as a quotient of the affine Hecke algebra of type $\tilde{A}_n$, using generators and relations.

0. INTRODUCTION

Brauer's centralizer algebra is defined abstractly as having a basis of diagrams as below, multiplied in a rather obvious fashion (see [B]) which involves a parameter δ. This algebra is an abstract model for the commutants of the tensor powers of the defining representations of (odd) orthogonal and symplectic groups, the parameter δ in the algebra being $\pm$ the dimension of the space. For generic values of the parameter the Brauer algebra is semisimple and its structure is known (see [W], [HW]).



A basis element of the Brauer algebra on four points.

The Brauer algebra on n points contains certain subalgebras defined by "topological" conditions. The most obvious is the so-called Temperley-Lieb

[1]) Partially supported by National Science Foundation grant DMS-91111411, U.S. Air Force grant F49620-92-J-0067, and the Swiss Fonds National.

algebra spanned by diagrams that are planar (have no crossings). It is a much smaller algebra than the Brauer algebra and its structure is easy and extremely well known, at least when semisimple (see [GHJ], [GW]). In this paper we analyse a slightly larger subalgebra of the Brauer algebra, namely that spanned by diagrams that can be realised without crossings in an annulus; see below.

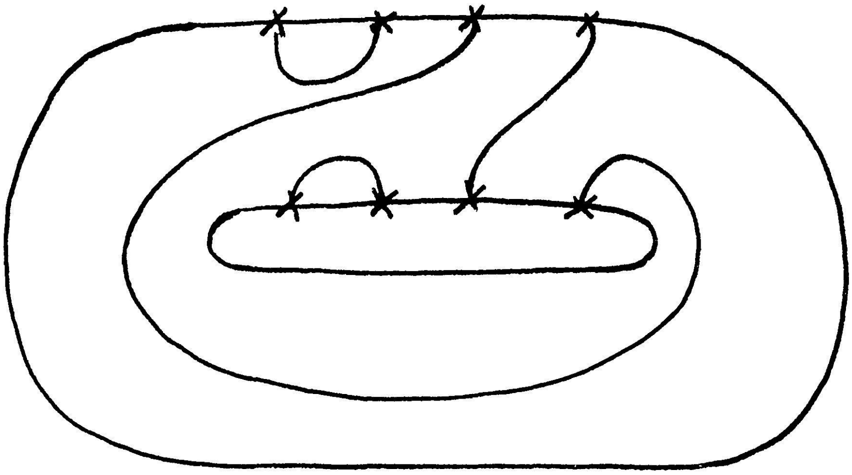


FIGURE 0: An annular diagram.

If we call β the Brauer representation into the commutant of $O(k)$ on $\otimes^n \mathbf{C}^k$, it is known that the restriction of β to the Temperley-Lieb algebra is faithful for $k \geqslant 2$. We show that the restriction to our annular algebra is faithful for $k \geqslant 3$ (but not for $k = 2$). This shows that the annular algebra is generically semisimple. For a value of δ for which the annular algebra is semisimple we show that the irreducible representations of the annular algebra (over $\mathbf{C}$) are parametrised by

(1) an integer t, $0 \leqslant t \leqslant n$ with $n + t$ even,

(2) a t-th root of unity.

The dimension of the corresponding representation, called $\pi_{t,\omega}$, is $\binom{n}{\frac{n-t}{2}}$ for $t > 0$ and $\frac{1}{\frac{n}{2}+1}\binom{n}{\frac{n}{2}}$ for $t = 0$ so that the dimension of the algebra is

$$\left\{\frac{1}{\frac{n}{2}+1}\binom{n}{\frac{n}{2}}\right\}^2 + \sum_{\substack{t+n \text{ even} \\ 0 < t \leqslant n}} t\binom{n}{\frac{n-t}{2}}^2,$$

which is the same as the number of annular diagrams.

There is a trace functional on the annular algebra which can be defined either via the Brauer representation or by counting the number of closed loops when a diagram is closed by identifying the inside and outside circles. This trace is determined by its values on idempotents $p_{t,\omega}$ whose principal left ideals define the representations $\pi_{t,\omega}$. Call the trace of such an idempotent $\mathcal{M}(\omega, t)$. Then we show that $t\mathcal{M}(\omega, t)$ is the integer valued polynomial in δ given by $2\sum_{r=0}^{t-1}\omega^r \cos(GCD(t, r)\theta)$, $2\cos\theta = \delta$. These also give the multiplicities of $\pi_{t,\omega}$ in β. In appendix 1 we give a table of the dimensions of the irreducible representations of the annular algebra for $n > 9$, and a table of the weights of the above trace.

The key to the analysis of the annular algebra is the observation that it is filtered by ideals corresponding to the number of "through-strings". This idea occurs in [B] (see [HW]) and we have taken the terminology from [MW]. For us it was inspired by the first way of counting annular diagrams presented in § 1, which was discovered by F. Jaeger, to whom the author is most grateful. It was necessary to use this "through string" technique, which is, technologically speaking, a backwards step from Wenzl's paper [W], since the annular algebras are *not* unitally included in one another. This means that the "basic construction" technique is not available.

A special system of generators of the annular algebra exhibits it as a quotient of the affine Hecke algebra of type $\tilde{A}_n$ with parameter $q(\delta = 2 + q + q^{-1})$ (see remarks after Theorem 2.8).

The original motivation for this work was to help calculate centraliser towers in subfactors. Given an extremal (see [PP]) subfactor N of a II_1 factor M, one forms the tower M_i as in [J] with $M_{i+1} = \langle M_i, e_{i+1}\rangle$, $M_0 = M$, $M_{-1} = N$. Then there is an action of an affine Hecke quotient on $N' \cap M_n$ according to the generators $f_1, f_2, \ldots, f_{2n+2}$ defined by:

$$f_i = \begin{cases} \text{left multiplication by } e_i, & 1 \leqslant i \leqslant n \\ E_{M_{n-1}} \text{ (conditional expectation)}, & \text{for } i = n+1 \\ \text{right multiplication by } e_{2n-i+2} & n+2 \leqslant i \leqslant 2n+1 \\ E_{M'}, & \text{for } i = 2n+2. \end{cases}$$

They satisfy: $f_i^2 = f_i$, $f_i f_{i\pm 1} f_i = \tau f_i$, $f_i f_j = f_j f_i$ if $j \neq i \pm 1$, where the indices are taken in $\mathbf{Z}/(2n+2)\mathbf{Z}$, and where τ^{-1} is the index of N in M. The result of this paper gives the structure of the algebra in the example $M = N \otimes M_k(\mathbf{C})$ — it is the oriented subalgebra of the annular algebra. In general the affine Hecke modules occurring in $N' \cap M_n$ depend sensitively on the subfactor. We will present more results on this elsewhere.

I would also like to thank D. Levy (see [Le]) and C. Cibils for conversations and S. Eliahou for some useful computer calculations.