# §3. Spinor zeta functions

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The following questions therefore are suggestive:

1) if one starts with an arbitrary $F \in M_{1/2, k-1/2}(\Gamma_2)$, does the above limit process produce skew-holomorphic Jacobi forms of weight $k$?

2) define $M^*_{1/2, k-1/2}(\Gamma_2)$ as the subspace of $M_{1/2, k-1/2}(\Gamma_2)$ consisting of the intersection of the kernels of the operators $\mathscr{C}_p$ for all primes $p$. Does there exist a natural map $V$ from skew-holomorphic Jacobi forms of weight $k$ and index 1 to $M^*_{1/2, k-1/2}(\Gamma_2)$ similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on $\mathrm{Sp}_2$. It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus $n > 2$ has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for $n \geqslant 33$ the map which sends a Siegel modular form of weight 16 on $\Gamma_n := \mathrm{Sp}_n(\mathbf{Z})$ to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for $n = 3$ due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform $F$ of even integral weight $k$ on $\Gamma_3$ could be constructed from a pair $(f, g)$ of elliptic Hecke eigenforms of weights $(k_1, k_2)$ equal to $(k, 2k-4)$ or $(k-2, 2k-2)$ such that the (formal) spinor zeta function of $F$ should be equal to $L_f(s - k_2/2) L_f(s - k_2/2 + 1) L_{f \otimes g}(s)$ where $L_{f \otimes g}(s)$ essentially is the Rankin convolution of $f$ and $g$ ([*loc. cit.*, §4]; note that for $n > 2$ the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on $\Gamma_n$ is not known).

## §3. Spinor zeta functions

### 3.1. Results

Although the Maass space $S^*_k(\Gamma_2)$ as discussed in the previous section is an important subspace of $S_k(\Gamma_2)$ in its own right, one quickly realizes that the "true" Siegel cusp forms on $\Gamma_2$ should lie in the orthogonal complement of $S^*_k(\Gamma_2)$ (cf. Theorem 2 in §2 and its discussion). Is is therefore even more surprising that forms in the Maass space can be used to study forms in $S_k^*(\Gamma_2)^\perp$ (in fact, spinor zeta functions of Hecke eigenforms in $S_k^*(\Gamma_2)^\perp$). Thus the importance of the Maass space seems to go much beyond that what is expected from §2.

Let $F$ and $G$ be Siegel cusp forms of integral weight $k$ on $\Gamma_2$. Denote by $\phi_m$ and $\psi_m (m \geqslant 1)$ the Fourier-Jacobi coefficients of $F$ and $G$, respectively and define a formal Dirichlet series of Rankin-type by

$$D_{F,G}(s) := \zeta(2s-2k+4) \sum_{m \geqslant 1} < \phi_m, \psi_m > m^{-s} \tag{6}$$

(this series was introduced by Skoruppa and the author in [18]).

A variant of the classical Hecke argument shows that $< \phi_m, \psi_m > \ll_{F,G} m^k$ so that $D_{F,G}(s)$ is absolutely convergent for $\mathrm{Re}(s) > k+1$. We put

$$D_{F,G}^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)D_{F,G}(s) \quad (\mathrm{Re}(s) > k+1) .$$

THEOREM 1 [18]. *The function $D_{F,G}(s)$ has a meromorphic continuation to* $\mathbf{C}$ *which is holomorphic except for a possible simple pole of residue*

$$\frac{4^k \pi^{k+2}}{(k-2)!} < F, G >$$

*at $s = k$. Furthermore, the functional equation*

$$D_{F,G}^*(2k-2-s) = D_{F,G}^*(s)$$

*holds.*

THEOREM 2 [18]. *Let $k$ be even. Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform and $G$ be a function in the Maass space $S_k^*(\Gamma_2)$. Then*

$$D_{F,G}(s) = < \phi_1, \psi_1 > Z_F(s) .$$

The proof of Theorem 1 is based on the Rankin-Selberg method applied with an Eisenstein series of Klingen-type on $\mathrm{Sp}_2$. The proof of Theorem 2 uses Theorem 1 of §2 applied with $\phi$ a Poincaré series; furthermore, an explicit formula for the action on Fourier coefficients of the operator $V_m^*$ adjoint to $V_m$ w.r.t the Petersson scalar products and the relations due to Andrianov [1, Chap. 2] between eigenvalues and Fourier coefficients of Hecke eigenforms play an important role. Let us mention that Theorem 2 could also be deduced from results of Gritsenko [13, p. 266].

In [38], Yamazaki using the theory of Eisenstein series à la Langlands studied the analytic properties of generalizations to arbitrary genus $n$ of the series (6). Recently, Krieg [24] gave a more elementary proof of (some of) the results of [38] using well-known properties of Epstein zeta functions. However, it is clear from the $\Gamma$-factors and the type of the functional equations that for $n > 2$ there cannot be any direct connection between the series studied in [24, 38] and spinor zeta functions.

1.2 PROBLEMS

i) Suppose that $k$ is even. If $F$ is a non-zero Hecke eigenform in $S_k(\Gamma_2)$, is $\phi_1 \neq 0$? (This question was already asked in [33].) The answer is positive for $k \leqslant 32$ as numerical computations due to Skoruppa [35] show. Note that by Theorem 2 a positive answer gives a new proof for the analytic continuation and the functional equation of $Z_F(s)$.

ii) Let $F$ be a Hecke eigenform in $S_k(\Gamma_2)$. The only critical point of $Z_F(s)$ in Deligne's sense is $s = k - 1$, i.e. the center of symmetry of the functional equation as is easily checked. Conjecturally therefore $Z_F(k-1)$ should be equal to the determinant of a "period matrix" times an algebraic number (one may suppose that $k$ is even since otherwise $Z_F(k-1) = 0$ as follows from the sign in the functional equation). To the author's knowledge, nothing so far in this direction has been proved. Could Theorem 2 eventually be useful in this context?

As a side remark, let us mention here that Böcherer [4] motivated by Waldspurger's results [37] about the central critical values of quadratic twists of Hecke $L$-functions of elliptic Hecke eigenforms, for $k$ even has conjectured that the central critical value of the twist of $Z_F(s)$ by a quadratic Dirichlet character of conductor $D < 0$ should be proportional to the *square* of $\sum_{\{T>0\}/\sim,\, \mathrm{disc}\, T = D} a(T)$ where $a(T)$ are the Fourier coefficients of $F$ and the sum is over a set of $\Gamma_1$-representatives of positive definite integral binary quadratic forms $T$ of discriminant $D$. This conjecture is true if $F$ is in the Maass space as follows from Theorem 2 in §2 in connection with Waldspurger's results, cf. [4]. The conjecture when generalized to level $> 1$ is also true if the corresponding form has weight 2 and is the Yoshida lift of two elliptic cusp forms [6].

iii) Let $F$ be a cuspidal Hecke eigenform and assume that $F$ is in $S_k^*(\Gamma_2)^\perp$ if $k$ is even. Does the function $D_{F,F}(s)$ have any intrinsic arithmetical meaning? (This question was already asked in [33], too; note that $D_{F,F}(s)$ for $F$ as above cannot be proportional to $Z_F(s)$ since $D_{F,F}(s)$ has a pole at $s = k$ while $Z_F(s)$ is holomorphic there, cf. §2). For some numerical computations in this direction in the case $k = 20$ (the first case where $S_k^*(\Gamma_2)^\perp \neq \{0\}$) we refer to [23].