§2. The root System D_4 and the Hurwitz quaternionic integers

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1.3. PROPOSITION. *A unimodular symmetric bilinear form S admits a decomposition $S = B + B'$ with B unimodular if and only if S has a perfect isometry.*

Thus, Question 1.1 reduces to the following.

1.4. QUESTION. Given a unimodular symmetric bilinear form S, does there exist a perfect isometry of S?

Note that if S is positive definite and even, then the rank of S is a multiple of 8. M. Kervaire gives a complete answer to Question 1.4, for positive definite forms of rank less than or equal to 24. For forms of arbitrary rank, he proves the following partial result, using the theory of the associated root systems.

Let $\mathrm{R} = \{x \in L \mid S(x, x) = 2\}$. Suppose that R is a root system in $\mathbf{R}^n$ of rank n $(= \operatorname{rank} L)$. Then the irreducible components of R are of type A, D, or E; and we have:

1.5. THEOREM ([K], Cor. 3, Prop. 4).

(a) *If* R *has an irreducible component of type* A_{2k-1}, E_7 *or* D_{k+4}, *with* $k \geqslant 1$, *then there does not exist any perfect isometry of* (L, S).

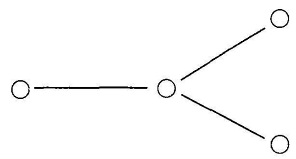
(b) *If* $\mathrm{R} = \underset{1 \leqslant i \leqslant p}{\perp} \mathrm{A}_{2k_i} \perp q\mathrm{E}_6 \perp r\mathrm{E}_8$, *then there exists a perfect isometry of* L, *inducing a perfect isomorphism of the abelian group* $\mathbf{Z}\mathrm{R}^{\#}/\mathbf{Z}\mathrm{R}$, *which corresponds to multiplication by* -1, *where* $\mathbf{Z}\mathrm{R}^{\#}$ *denotes the dual of the lattice* $\mathbf{Z}\mathrm{R}$.

Note that the case of R having an irreducible component of type D_4 is not covered by this theorem. In this paper we give an analogue of (b) for this case. In fact, we first consider the case in which R is of type $n\mathrm{D}_4$. In this case, we show (Th. 5.2) that (L, S) admits a perfect isometry if and only if the isometry class of (L, S) contains a symmetric bilinear space (L', S') of some hermitian space over the Hurwitz quaternionic integers. The analogue of Proposition 1.5 follows from this immediately (Theorem 5.3). In the final section we also give some examples.

§2. THE ROOT SYSTEM D_4 AND THE HURWITZ QUATERNIONIC INTEGERS

The fact that the root lattice $\mathbf{Z}\mathrm{D}_4$ can be identified with the lattice of Hurwitz quaternionic integers was long recognized: see for instance ([C-S]). However we give here a direct proof of this fact and recall some arithmetical facts about these quaternionic integers, which are needed in the sequel.

We first fix the following terminology. By a **Z**-*lattice* we mean a pair (L, b), where L is a finitely generated free **Z**-module and $b: L \times L \to \mathbf{Z}$ a positive definite, even, symmetric bilinear form. If the set $\{x \in L \mid b(x, x) = 2\}$ forms a root system of type $n\mathrm{D}_4$ where the rank of L equals $4n$, then we call it a **Z**-*lattice of type* $n\mathrm{D}_4$. If L is contained in $\mathbf{R}^m$ and b is induced by the Euclidean inner product on $\mathbf{R}^m$, we call it a *Euclidean* **Z**-*lattice*. The symbol D_4 will always mean the root system in $\mathbf{R}^4$ with the Euclidean inner product, corresponding to the Dynkin diagram



Let $\mathscr{A} = \mathbf{Q} \oplus \mathbf{Q}i \oplus \mathbf{Q}j \oplus \mathbf{Q}k$ denote the quaternion division algebra over **Q**, defined by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k\,.$$

Let $h: \mathscr{A}^n \times \mathscr{A}^n \to \mathscr{A}$ be the hermitian form defined by

$$h((x_1 \dots x_n), (y_1, \dots, y_n)) = \sum_1^n x_i \bar{y}_i\,,$$

where bar denotes the conjugation in $\mathscr{A}$. If $Tr: \mathscr{A} \to \mathbf{Q}$ denotes the trace map $Tr(x) = x + \bar{x}$, then $Tr \circ h$ is a positive definite symmetric bilinear form over **Q**. Let $\mathscr{H}$ denote the Hurwitz quaternionic integers in $\mathscr{A}$ i.e. $\mathscr{H} = \{(a + bi + cj + dk)/2 \mid a, b, c, d \in \mathbf{Z}$, with the same parity$\}$. Then, $\mathscr{H}$ is a maximal order in $\mathscr{A}$ and $(\mathscr{H}, Tr \circ h)$ is a **Z**-lattice. It is trivial to verify that $\xi_1 = (1 + i + j + k)/2$, $\xi_2 = (1 + i + j - k)/2$, $\xi_3 = (1 + i - j + k)/2$, and $\xi_4 = (1 - i + j + k)/2$ form a **Z**-basis of $\mathscr{H}$. Let $\mathscr{H}^*$ denote the dual of $\mathscr{H}$ in $\mathscr{A}$. Then we have

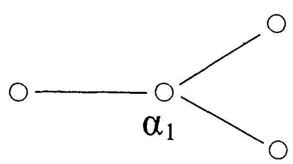
2.1. PROPOSITION.

(a) *The* **Z**-*lattice* $(\mathscr{H}, Tr \circ h)$ *is isometric to the Euclidean lattice* $\mathbf{Z}\mathrm{D}_4$.

(b) *The group of units of* $\mathscr{H}$ *forms a root system isomorphic to* D_4.

(c) *Every* **Z**-*lattice of type* $n\mathrm{D}_4$ *is isometric to a* **Z**-*lattice* L *such that* $\mathscr{H}^n \subset L \subset \mathscr{H}^{*n}$, *where the bilinear form on* L *is induced by* $Tr \circ h$.

Proof. Let $\{\varepsilon_i\}$ denote the standard orthonormal basis in $\mathbf{R}^4$, and let $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_1 - \varepsilon_2$, $\alpha_3 = \varepsilon_3 - \varepsilon_4$, $\alpha_4 = \varepsilon_3 + \varepsilon_4$. Then $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis for the root system D_4. The associated Dynkin diagram is given by



If b denotes the Euclidean inner product on $\mathbf{R}^4$, then, $\tau: \mathscr{H} \to \mathbf{Z}\mathrm{D}_4$ defined by $\tau(\xi_1) = \alpha_1$, $\tau(\xi_i) = -\alpha_i$, $2 \leqslant i \leqslant 4$, is an isometry of $(\mathscr{H}, Tr \circ h)$ onto $(\mathbf{Z}\mathrm{D}_4, b)$. This proves (a). We note that an element x in $\mathscr{H}$ is a unit if and only if $Tr \circ h(x) = 2$. Hence (b) follows from the above isometry. Since $Tr \circ h$ is nondegenerate, the dual of $\mathscr{H}$ in $\mathscr{A}$ is the same as the dual of $\mathscr{H}$ in $\mathscr{A} \otimes_{Q} \mathbf{R} \simeq \mathbf{R}^4$. From (a) it follows that $\mathscr{H}^*$ is isometric to $(\mathbf{Z}\mathrm{D}_4)^{\#}$. Thus (c) follows from the fact that every $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$ is isometric to a Euclidean $\mathbf{Z}$-lattice L such that $(\mathbf{Z}\mathrm{D}_4)^n \subset L \subset (\mathbf{Z}\mathrm{D}_4^{\#})^n$.

Let us now recall a few arithmetical facts about the Hurwitz quaternionic integers, details of which can be found in [R]. The dual $\mathscr{H}^*$ is a two-sided full $\mathscr{H}$-module in $\mathscr{A}$ i.e. an $\mathscr{H}$-submodule of $\mathscr{A}$ which contains a $\mathbf{Q}$-basis of $\mathscr{A}$. The set of all two-sided full $\mathscr{H}$-submodules of $\mathscr{A}$ is a free abelian group with the set of all maximal ideals of $\mathscr{H}$ as basis. Further the inverse of $\mathscr{H}^*$ is a maximal ideal in $\mathscr{H}$. In fact, $(\mathscr{H}^*)^{-1} = \mathscr{P}$, $\mathscr{P} = (1+i)$, $\mathscr{P}^2 = (2)$, $\mathscr{P} = \bar{\mathscr{P}}$, and $\mathscr{H}/\mathscr{P} \simeq \mathbf{F}_4$. We have,

2.2. Proposition.

(a) *The quotient* $\mathscr{H}^*/\mathscr{H}$ *has the natural structure of a vector space of dimension one over* $\mathbf{F}_4$.

(b) *The hermitian form* h *induces a hermitian form* $\eta(h)$ *on* $\mathscr{H}^*/\mathscr{H}$, *with values in* $\mathscr{H}^{*2}/\mathscr{H}^*$, *which is isometric to the standard hermitian form on* $\mathbf{F}_4$.

Proof. (a) This follows from the fact that, $\mathscr{H}^*$ is an $\mathscr{H}$-module of rank one and $\mathscr{P}\mathscr{H}^* = \mathscr{H}^*\mathscr{P} = \mathscr{H}$.

(b) This follows from the commutativity of the diagram:

$$\begin{array}{ccc} \mathscr{H}^*/\mathscr{H} \times \mathscr{H}^*/\mathscr{H} & \to & \mathscr{H}^{*2}/\mathscr{H}^* \\ \downarrow & & \downarrow \\ \mathscr{H}/\mathscr{P} \times \mathscr{H}/\mathscr{P} & \to & \mathscr{H}/\mathscr{P} \end{array}$$

where the vertical arrows are the isomorphisms induced by multiplication by $1+i$ and 2 respectively and the horizontal arrows are the respective hermitian forms.

From now on, we shall identify $\mathscr{H}^*/\mathscr{H}$ with $\mathbf{F}_4$, as a one dimensional vector space for the choice of the basis $1/1+i$.

2.3. PROPOSITION.

(a) *Let* $\mathscr{H}^n \subseteq L \subseteq \mathscr{H}^{*n}$ *be a* $\mathbf{Z}$*-module. Then* $(L, Tr \circ h)$ *is integral if and only if* $\eta(L)$ *is a totally isotropic subspace of the symmetric bilinear space* $(\mathbf{F}_4^n, Tr \circ \eta(h))$, *where* $\eta(h)$ *is the standard hermitian form on* $\mathbf{F}_4^n$.

(b) *The* $\mathbf{Z}$*-lattice* $(L, Tr \circ h)$ *is unimodular if and only if* $\eta(L)$ *is a maximal totally isotropic subspace of* $(\mathbf{F}_4^n, Tr \circ \eta(h))$.

Proof. (a) This follows easily from 2.2.

(b) This follows from (a), since L is unimodular if and only if L is maximal integral.

§3. PERFECT ISOMETRIES OF $\mathscr{H}$-LATTICES

In this section we show that certain special class of $\mathbf{Z}$-lattices admit perfect isometries. We begin with the following definition.

3.1. *Definition.* A $\mathbf{Z}$-lattice (L, b) is called an *$\mathscr{H}$-lattice* if L is an $\mathscr{H}$-module and $b = Tr \circ h$ for some hermitian form h.

3.2. PROPOSITION. *Every $\mathscr{H}$-lattice has a perfect isometry.*

Proof. Let $(L, Tr \circ h)$ be an $\mathscr{H}$-lattice. Let $\sigma : L \to L$ denote left (or right) multiplication by ξ where ξ is one of the units $(1 \pm i \pm j \pm k)/2$. Then,

$$Tr \circ h(\sigma(x), \sigma(y)) = Tr \circ h(\xi x, \xi y) = Tr(\xi h(x, y)\bar{\xi})$$
$$= \xi h(x, y)\bar{\xi} + \xi\overline{h(x, y)}\bar{\xi} = \xi(h(x, y) + \overline{h(x, y)})\,\bar{\xi} = \xi\bar{\xi}(h(x, y) + \overline{h(x, y)})$$
$$= h(x, y) + \overline{h(x, y)} = Tr \circ h(x, y)\ .$$

Therefore σ is an isometry. Since the minimal polynomial of σ is $x^2 - x + 1$, $\det(1 - \sigma) = 1$ and hence σ is perfect.

As a special case of this we have:

3.3. COROLLARY. *The $\mathscr{H}$-lattice $(\mathscr{H}, Tr \circ h)$ has a perfect isometry.*

3.4. PROPOSITION. *Every perfect isometry of $(\mathscr{H}, Tr \circ h)$ induces a perfect $\mathbf{F}_2$-isomorphism of $\mathscr{H}^*/\mathscr{H} = \mathbf{F}_4$, which corresponds to multiplication by ω, where $\mathbf{F}_2(\omega) = \mathbf{F}_4$.*