6. Graphs, computations, and the shape of $\bar{W}$

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The computations of zeros graphed in our figures were performed in double precision (approx. 18 decimal places) on a Silicon Graphics workstation. Some of the zeros were checked for accuracy by recomputing them in double precision (approx. 28 decimal places) on a Cray X-MP. The zero-finding program used the Jenkins-Traub algorithm and was taken from a standard subroutine library. Checks showed that the values that were obtained were accurate on average to at least 10 decimal places, which was sufficient for our graphs. The program that was used appeared to produce accurate values on the Cray for the zeros for polynomials of degrees up to about 150. (Computation of zeros of polynomials of much higher degree would have required better algorithms, cf [9].)

Zeros of a large set of random polynomials $f(z) \in P$ of degree 100 were computed on the Cray, and they exhibit most of the features visible in Figures 1-3. However, they are not as interesting as the lower degree zeros that are exhibited in Figures 1-3. The "spikes" or "tendrils" that generate the fractal appearance in the graphs we include come from a small fraction of the polynomials. Sampling even 10^4 of the 2^{99} polynomials $f(z) \in P$ of degree 100 does not yield a good representation of the extremal features that we expect to see for high as well as low degrees.

Graphs were prepared using the S system [2].

The graphs in Figures 4-6 were prepared differently. A program was written that checked whether a given w with $|w| < 1$ is in $\bar{W}$. Note that

$$\left| \sum_{k=0}^{\infty} a_k w^k \right| \leqslant B = \max(1, |1 + w|)/(1 - |w|^2), \tag{6.1}$$

where the a_k are any 0, 1 coefficients, since we can write

$$\sum_{k=0}^{\infty} a_k w^k = (a_0 + a_1 w) + (a_2 + a_3 w) w^2 + \cdots .$$

The procedure was to test all sets of 0, 1 coefficients $a_1, \ldots, a_{120}$ to see whether they could be the initial segment of coefficients of a power series

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \tag{6.2}$$

for which $f(w) = 0$. Let us regard the strings of coefficients $a_1, \ldots, a_{120}$ as the leaves of a balanced binary tree, with the nodes below the root corresponding to a_1, those below to a_1, a_2, etc. The procedure was to explore this tree, checking whether

$$\left|1 + \sum_{j=1}^{d} a_j w^j\right| > |z|^{d+1} B \tag{6.3}$$

at any stage. If (6.3) is satisfied, then w is not a zero of any power series of the form (6.2) with initial coefficients $1, a_1, \ldots, a_d$, and the subtree of that node does not have to be explored. If all the leaves are discarded by this procedure, we have a rigorous proof that $w \notin \overline{W}$, and so in fact an open neighborhood of w is outside $\overline{W}$. On the other hand, if a leaf was found with

$$\left|1 + \sum_{j=1}^{120} a_j w^j\right| < |z|^{121} B/10 , \tag{6.4}$$

then the program assumed that $w \in \overline{W}$. (By establishing lower bounds for the derivative of the polynomial $1 + \sum_{1}^{120} a_j z^j$ at w and using crude upper bounds for the second derivative, one could in principle prove that there is some point w' close to w such that $w' \in \overline{W}$, although the 10 in condition (6.4) might have to be decreased. Another way to prove this would be to use Lemma 3.1. This step was not carried out.) Figures 4-6 were produced by testing each w in a 1936×1936 or a 1944×1944 grid (corresponding to the resolution of our laser printer). There were few points w for which neither condition (6.3) nor condition (6.4) held. The exceptions occur primarily in Figure 4, but they do not affect how the picture looks. Had we used a tree of depth 80, the exceptions would have been much more frequent.

The computations of Figures 4-6 are not completely rigorous in that the determination of $w \notin \overline{W}$ is rigorous, while that of $w \in \overline{W}$ is not. Moreover, an implicit premise in the preparation of Figures 4-6 was that if a point $w \in \overline{W}$, then the whole neighborhood of w represented by the corresponding pixel is in $\overline{W}$. On the other hand, the computations of Figures 1-3 are rigorous.

It is possible to use computations to obtain rigorous estimates for $\overline{W}$ that are sharper than those of Theorem 2.1. As an example, we sketch how a moderate amount of straightforward computing establishes that there are no $w \in \overline{W} \setminus \mathbf{R}$ with $|w| < 0.7$. We modify the method of proof of Theorem 2.1. Write

$$f(z) = 1 + \sum_{j=1}^{10} a_j z^j + \frac{1}{2}(z^{11} + z^{12} + \cdots) + g(z) , \tag{6.5}$$

where

$$g(z) = \frac{1}{2} \sum_{k=11}^{\infty} \varepsilon_k z^k , \qquad \varepsilon_k = \pm 1 . \tag{6.6}$$

Then we can write

$$f(z) = F(z) + g(z) ,$$

$$F(z) = G(z)/2(1 - z) ,$$

$$G(z) = 2(1 - z) \left(1 + \sum_{j=1}^{10} a_j z^j\right) + z^{11} .$$

If we establish that $|F(z)| > |g(z)|$ on some simple closed contour about the origin, then by Rouché's theorem $f(z)$ and $F(z)$ will have the same number of zeros inside that contour. To prove that $|F(z)| > |g(z)|$ on a contour C, it suffices to show that $|F(z)| > g(z) + \delta$ on a discrete set of points z on C, where $\delta > 0$ is such that bounds on the derivatives of $F(z)$ and $g(z)$ guarantee that $|F(z)| - |g(z)|$ will not decrease by more than δ between the sampling points. This was applied to each of the 2^{10} choices of $a_1, \ldots, a_{10}$. Of the 1024 functions $F(z)$, 997 satisfied $|F(z)| > |g(z)|$ on

$$C_3 = \{z : |z| = 0.7\} .$$

The remaining 27 functions $F(z)$ were shown to satisfy $|F(z)| > |g(z)|$ on the contour

$$\begin{aligned} C_4 = \{z : |z| = 0.7, \quad |y| \geqslant 0.04\} \\ \cup \{z : x = -0.74, |y| \leqslant 0.04\} \\ \cup \{z : |y| = 0.04, \ -0.74 \leqslant x \leqslant -0.6, |z| \geqslant 0.7\} . \end{aligned}$$

Finally, zeros of each of the 1024 polynomials $G(z)$ were computed, and it was found that 85 of these polynomials had a single zero in $|z| \leqslant 0.74$, and the remaining 939 had none. Thus in all cases we can conclude that $f(z)$ has at most one zero in $|z| \leqslant 0.7$. Such a zero has to be real.

The estimates used above were crude, and with more care one can either decrease the amount of computing (and even eliminate it altogether) or obtain better bounds for $\bar{W}$.

The basic principle that makes it possible to obtain good estimates of $\bar{W}$ is that for extremal points $w \in \bar{W}$, the power series $f(z)$ with 0, 1 coefficients such that $f(w) = 0$ are restricted. For example the region depicted in Figures 3 and 4 is

$$V = \{z = x + iy : -0.501 \leqslant x \leqslant -0.497, 0.537 \leqslant y \leqslant 0.541\} .$$

Numerical computation (evaluating polynomials of degrees $\leqslant 9$ with 0, 1 coefficients at a 41×41 uniform grid, and bounding derivatives) shows that if $w \in V \cap \bar{W}$, then w can only be a zero of a power series of the form

$$f(z) = 1 + z + z^2 + z^4 + z^7 + z^9 + \sum_{k=10}^{\infty} a_k z^{10} .$$

This restricted the set of $f(z)$ that had to be considered, and made possible the computation of Figure 3, as it would not have been feasible to examine all polynomials of degrees $\leqslant 32$. Furthermore, this restriction on the coefficients of $f(z)$ makes it possible to estimate the shape of $V \cap \bar{W}$.

It should be possible to prove rigorously, with the help of numerical computations, such as those mentioned above, that the hole in $\bar{W}$ mentioned in the Introduction and pictured in Figure 6 is isolated in the sense that there is a continuous closed curve in $\bar{W} \cap U$, for U a small rectangle, that encloses the hole. We have not done this.

To explain the fractal appearance of $\bar{W}$, suppose that $w \in W$, $|w| < 1$, and that $f(w) = 0$ where

$$f(z) = 1 + \sum_{j=1}^{d} a_j z^j , \qquad a_j = 0, 1 .$$

Suppose that

$$g(z) = f(z) + z^{d+1} \sum_{k=0}^{\infty} b_k z^k , \qquad b_k = 0, 1 .$$

If $g(z) = 0$ and $|z - w|$ is small, while d is large, we have

$$\begin{aligned} 0 = g(z) &\cong g(w) + (z - w) g'(w) \\ &\cong w^{d+1} \sum_{k=0}^{\infty} b_k w^k + (z - w) f'(w) . \end{aligned}$$

If $f'(w) \neq 0$ (which as far as we know may hold for all w with $|w| < 1$), then $g'(w) \neq 0$ for d large enough, and we can expect that

$$z \cong w - \frac{w^{d+1} \sum_{k=0}^{\infty} b_k w^k}{f'(w)} .$$

Thus if

$$Q(w) = \left\{ \sum_{k=0}^{\infty} b_k w^k : b_k = 0, 1 \right\} ,$$

then we expect to find zeros in a neighborhood of each point of

$$w - w^{d+1}(f'(w))^{-1}Q(w) .$$

The set $Q(w)$ is connected [1], and for $w \notin \mathbf{R}$, it seems that it contains a small disk around the origin. The set $Q(w)$ is a continuous function of w, which accounts for the similarity of the protrusions from $\overline{W}$ visible in Figures 5 and 6. (The protrusions in Figure 4 are different, since there the sets $Q(w)$ are of different shape from those in Figures 5 and 6.)

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