

# 3. A COMBINATORIAL INTERLUDE

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COROLLARY 2.9. *As representations of  $Y$ , we have*

$$W_m(a)^* \cong W_m(-a).$$

*Proof.* On  $W_m(a)$ ,  $J(x)$  acts as  $ax$ . Therefore, on  $W_m(a)^*$ ,  $J(x)$  acts as  $-ax$ .

The following is a related result.

PROPOSITION 2.10. *Every evaluation representation  $W_m(a)$  has a non-degenerate invariant symmetric bilinear form.*

This means that there is a non-degenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $W_m(a)$  such that

$$(2.11) \quad \langle y \cdot v_1, v_2 \rangle = \langle v_1, \omega(y) \cdot v_2 \rangle$$

for all  $y \in Y$ ,  $v_1, v_2 \in W_m(a)$ .

*Proof.* It is well-known that the representation  $W_m$  of  $\mathfrak{sl}_2$  carries a form  $\langle \cdot, \cdot \rangle$  which satisfies (2.11) for all  $y \in \mathfrak{sl}_2$ . Moreover, the form is unique up to a scalar multiple because  $W_m$  is irreducible. To prove (2.11) in general, it suffices to check the case  $y = x_k^+$ , since the case  $y = x_k^-$  then follows because  $\langle \cdot, \cdot \rangle$  is symmetric, and  $\omega(x_k^+) = x_k^-$ . Since vectors of different weights are orthogonal, it is therefore enough to prove.

$$(2.12) \quad \langle x_k^+ \cdot e_i, e_{i+k} \rangle = \langle e_i, x_k^- \cdot e_{i+k} \rangle$$

(with the understanding that  $e_i = 0$  unless  $0 \leq i \leq n$ ). This follows easily from Proposition 2.6 and the invariance of  $\langle \cdot, \cdot \rangle$  under  $\mathfrak{sl}_2$ .

### 3. A COMBINATORIAL INTERLUDE

The form of the polynomial  $P$  associated to the representation  $W_m(a)$  in Corollary 2.7(b) suggests the following definition.

*Definition 3.1.* A non-empty finite set of complex numbers is said to be a *string* if it is of the form  $\{a, a+1, \dots, a+n\}$  for some  $a \in \mathbf{C}$  and some  $n \in \mathbf{N}$ .

The centre of the string is  $a + \frac{n}{2}$  and its length is  $n+1$ .

We shall also need:

*Definition 3.2.* Two strings  $S_1$  and  $S_2$  are said to be *non-interacting* if either

- (1)  $S_1 \cup S_2$  is not a string, or
- (2)  $S_1 \subseteq S_2$  or  $S_2 \subseteq S_1$ .

*Remark.* We shall discuss the “interactions” of strings in section 4.

We should like to assert that the set of roots of an arbitrary polynomial is a union of non-interacting strings. To make this precise, we need one last definition.

*Definition 3.3.* A set with multiplicities is a map  $f : \Sigma \rightarrow \mathbf{N}$ , where  $\Sigma$  is a set. If  $\Sigma$  is a finite set, the cardinality of  $f$  is

$$|f| = \sum_{x \in \Sigma} f(x).$$

The union of two sets with multiplicities is the sum of the corresponding maps. Note that any set is a set with multiplicities, all values of the map being equal to one. Also, the roots of a polynomial  $P \in \mathbf{C}[u]$  form a set with multiplicities in a natural way. In particular, the roots of the polynomial associated to  $W_m(a)$  in Corollary 2.7(b) form a single string

$$S_m(a) = \left\{ a - \frac{1}{2}m + \frac{1}{2}, \dots, a + \frac{1}{2}m - \frac{1}{2} \right\}$$

with centre  $a$  and length  $m$ .

We shall need the following simple result whose verification we leave to the reader.

LEMMA 3.4. Two strings  $S_m(a)$  and  $S_n(b)$  are non-interacting if and only if it is not true that

$$|a - b| = \frac{1}{2}(m + n), \frac{1}{2}(m + n) - 1, \dots, \text{ or } \frac{1}{2}|m - n| + 1.$$

The result we want is:

PROPOSITION 3.5. Any finite set of complex numbers with multiplicities can be written uniquely as a union of strings, any two of which are non-interacting.

*Proof.* Let  $f : \Sigma \rightarrow \mathbf{N}$  be a finite set of complex numbers with multiplicities. The proof is by induction on  $|f|$ . If  $|f| = 0$  or 1 there is nothing to prove.

Choose  $s \in \Sigma$ , let  $S$  be the maximal string of numbers in  $\Sigma$  which contains  $s$ , and let  $g$  be the characteristic function of  $S$ . By induction,  $f - g$  is a union of non-interacting strings. If  $T$  is any such string, then  $S$  and  $T$  are non-interacting, since if  $T \not\subseteq S$  then  $S \cup T$  cannot be a string, by maximality of  $S$ . Thus, adjoining  $S$  to the string decomposition of  $f - g$  gives the desired decomposition of  $f$ .

As for uniqueness, we first show that the string  $S$  above must occur in any decomposition of  $f$  as a union of non-interacting strings. For, otherwise, let  $T$  be a maximal string in such a decomposition which contains  $s$ . Then  $T$  is properly contained in  $S$ , so there exists  $u \in S - T$  such that  $T \cup \{u\}$  is a string. Let  $U$  be a string in the given decomposition of  $f$  which contains  $u$ . Then, by its maximality,  $T$  cannot be contained in  $U$ , so  $T$  and  $U$  are interacting, a contradiction.

Thus,  $S$  must occur in any two decompositions of  $f$  as a union of non-interacting strings. Deleting  $S$  from both decompositions and using the induction hypothesis, one deduces that the two decompositions are the same.

We conclude this section with the computation of a determinant which plays the same role for Yangians as the Vandermonde determinant plays in the classification of integrable representations of affine Lie algebras [1].

Let  $r$  be a positive integer and let  $b_j, m_j, 1 \leq j \leq r$ , be complex numbers. Quantities  $d_{k,j}, A_{k,j}$  for  $1 \leq j \leq r, 0 \leq k \leq r - 1$ , are defined inductively by the following formulas:

$$(3.6) \quad \begin{aligned} A_{k,j} &= b_j^k + b_j^{k-1} d_{0,j} + \cdots + d_{k-1,j} \\ d_{k,j} &= m_{j+1} A_{k,j+1} + d_{k,j+1}, \quad d_{k,r} = 0 \end{aligned}$$

(we set  $d_{k,r+1} = 0$ ). Let  $A$  be the matrix  $(A_{k,j})$  with  $1 \leq j \leq r, 0 \leq k \leq r - 1$ .

**PROPOSITION 3.7.**  $\det A = \prod_{1 \leq k < j \leq r} (b_j - b_k - m_j)$ .

*Remark.* One can think of  $\det A$  as a ‘‘quantum Vandermonde determinant’’. Indeed, recall that  $Y$  is obtained from a deformation of  $U(\mathfrak{sl}_2[t])$  by setting the deformation parameter  $h$  equal to one. If we had not set  $h = 1$ , then in equation (3.6)  $d_{k,j}$  would be replaced by  $hd_{k,j}$  and in equation (3.7)  $m_j$  would be replaced by  $hm_j$ . Thus, in the ‘‘classical limit’’  $h \rightarrow 0$ ,  $\det A$  becomes the usual Vandermonde determinant and (3.7) its well-known factorization.

Our proof of (3.7) is rather indirect and will be given in the next section.