2. Goeritz matrices and the F-polynomial

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Now if we adopt the usual convention (see [5]) that the value of $\sigma_L(\omega)$ at a root of the Alexander polynomial is defined to be the mean of its two "adjacent" values

$$(18) \qquad \lim_{\varepsilon \to 0+} \sigma_L(\omega e^{i\varepsilon}) \quad \text{and} \quad \lim_{\varepsilon \to 0-} \sigma_L(\omega e^{i\varepsilon}),$$

the fact that both of these values are well-defined broad oriented skein invariants completes the proof that

COROLLARY 3. *The signature function* $\sigma_L : S^1 \to Z$ *is a broad oriented skein invariant for all links with non-zero Alexander polynomials.* □

This is an intriguing result, especially in view of the fact that $\sigma_L(\omega)$ is known to be a concordance invariant. It is natural to ask what relations there may be between skein theory and concordance theory. Another obvious question is that of what happens when the Alexander polynomial Δ_L is identically zero. In these circumstances the first Alexander ideal of the link collapses and the signature function can be thought of as extracting information about higher Alexander ideals. Kanenobu ([8] and [9]) has shown that there exist infinitely many links with identical P-polynomials but distinct second Alexander ideals, so there is no obvious reason to suppose that this information should be skein invariant. However, I know of no counter-examples to the conjecture that $\sigma_L(\omega)$ is a broad oriented skein invariant for all links.

2. GOERITZ MATRICES AND THE F-POLYNOMIAL

In this section I explore the relationships between the graph of a link, its Goeritz matrix and Kauffman's polynomial invariant $F_L(a, z)$. In particular I show that the $F(a, z)$, is essentially calculable from the Goeritz matrix of a knot. This result makes use of facts about planar graphs discovered by Whitney over 50 years ago.

2.1. THE GOERITZ MATRIX AND GRAPH OF A LINK

Kauffman [10] has defined a polynomial invariant $F_L(a, z)$ of oriented links as follows:

Recall the definition of the three *Reidemeister moves,* see Figure 3.

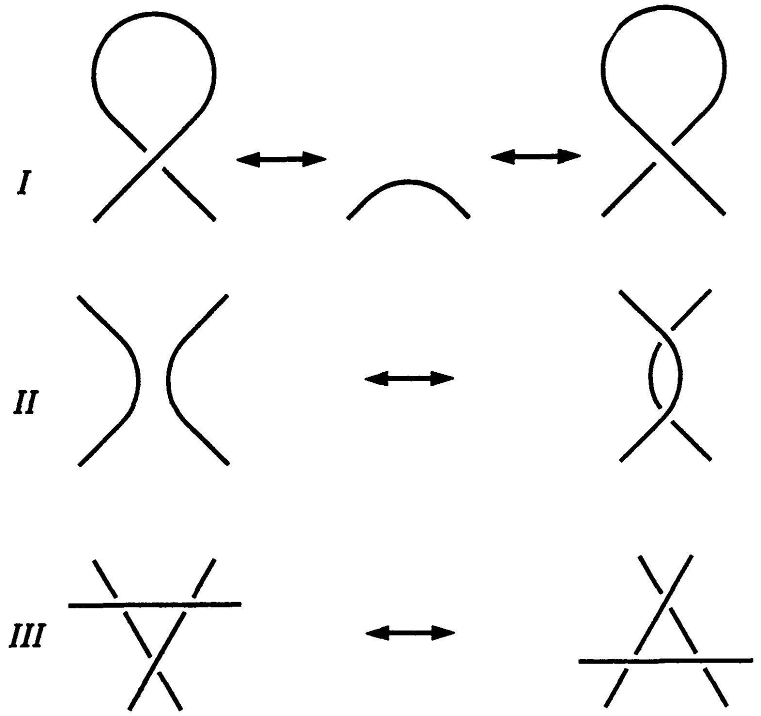

FIGURE 3

Two link diagrams represent the same link if and only if one can be transformed into the other by a finite sequence of these moves (see [18]). We define a polynomial invariant $\Lambda \in Z[a^{\pm 1}, z^{\pm 1}]$ of unoriented link diagrams by the four axioms:

i) Λ (unknot) $= 1$.

ii) Λ is invariant under Reidemeister moves II and III.

iii) The effect of Reidemeister move I on Λ is to multiply by a or a^{-1}:

$$\Lambda(\text{kink}_1) = a^{-1}\Lambda(\frown), \qquad \Lambda(\text{kink}_2) = a\Lambda(\frown). \tag{19}$$

iv) If four link diagrams L_+, L_-, L_0 and L_∞ are identical except in a ball B where they are as shown in figure 4 then

$$\Lambda(L_+) + \Lambda(L_-) = z\big(\Lambda(L_0) + \Lambda(L_\infty)\big) \tag{20}$$

Axioms i)-iv) are sufficient to define Λ for all link diagrams. Now given an oriented diagram we can temporarily forget its orientation and calculate its Λ-polynomial. Let w be the *writhe* of the diagram (that is, the number of positive crossings less the number of negative crossings). Then

$$F(a, z) = \Lambda(a, z) \cdot a^{-w} \tag{21}$$

is a link invariant, the *Kauffman polynomial* (see [10]). Note that in order to define F_L we need the writhe, which is orientation-dependant, so F_L is an invariant of oriented links. However, for knots (1-component links), reversing the orientation leaves the sign of any given crossing, and hence the writhe, unchanged, so for knots F_L may be regarded as an unoriented invariant. If $L = \bigcup_{i=1}^{n} L_i$ is an arbitrary oriented link with components L_i, then $F_L(a, z) \cdot a^{\lambda/2}$, where $\lambda = \sum_{i \neq j} lk(L_i, L_j)$ is the *total linking number of* L is unchanged by reversal of orientations of components, and so should be regarded as an unoriented link invariant (This observation has also been made by Turaev in [23]). Like $P_L(l, m)$, $F_L(a, z)$ behaves nicely with respect to disjoint and connected sums of links:

$$F_{L_1 \; L_2}(a, z) = F_{L_1}(a, z) \cdot F_{L_2}(a, z) \tag{22}$$

$$F_{L_1 \cup L_2}(a, z) = z^{-1}(a + a^{-1} - 1) \cdot F_{L_1}(a, z) \cdot F_{L_2}(a, z) \tag{23}$$

and is also invariant under mutation (see [1], [10]).

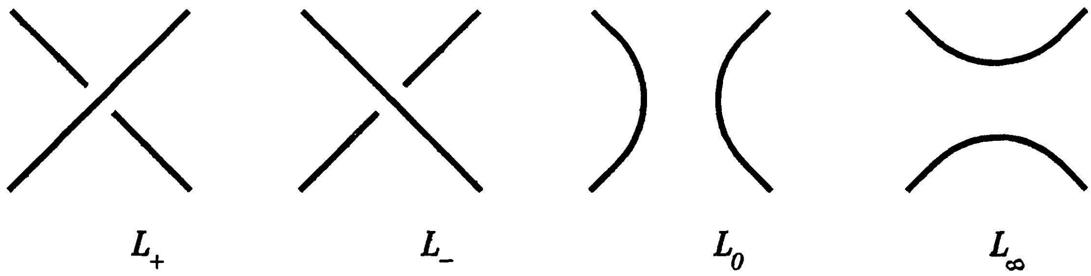

FIGURE 4

Now recall the definition of the *Goeritz matrix* of an unoriented connected link diagram $\mathscr{D}$ in the plane (see [6]). Such a diagram divides the plane into regions, which we proceed to colour black and white, chess board fashion, so that adjacent regions are distinct colours (It is not hard to see, using the Jordan curve theorem, that this can always be done). By convention we colour the infinite region white. Now label the black regions $R_1, R_2, \ldots, R_n$ say. At each crossing in the diagram a region R_i meets a region R_j, not necessarily distinct. This crossing takes one of two forms, illustrated in Figure 5, and we allocate signs $\xi = \pm 1$ to the crossings accordingly (The value ξ, which is defined only in the presence of a chess-board colouring of a link diagram, should not be confused with the sign of a crossing as defined in section 1. Unfortunately, the word "sign" is now well-established for each of these values in the literature.)

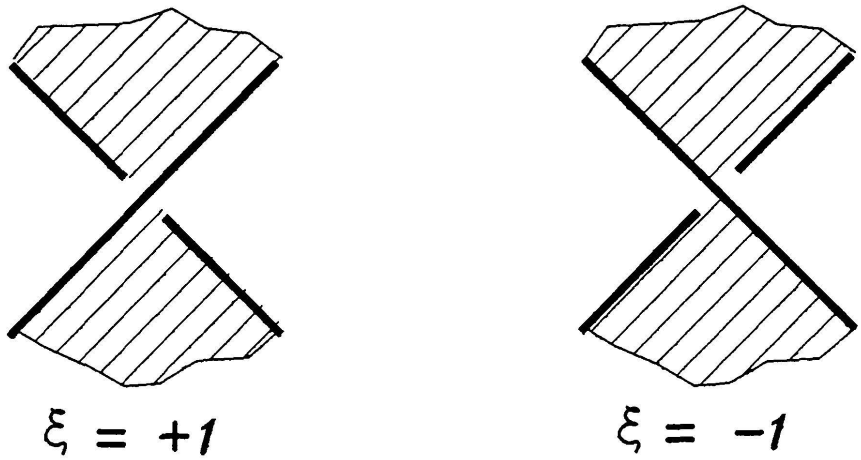

FIGURE 5

Now construct an $(n \times n)$ matrix A as follows: for $i \neq j$, let $a_{ij} = \sum \xi(c)$, where the sum is over all crossings c at which R_i meets R_j in the diagram. The diagonal elements are given by $a_{ii} = -\sum_{i \neq j} a_{ij}$ so that the row and column sums are all zero. A Goeritz matrix G for the link is then obtained by discarding the first row and column of A. Clearly G is not an invariant of the link, or even of the link diagram (any other row or column could have been discarded instead of the first one, for example). It is, however, a relation matrix for $H_1(D_L)$, where D_L is the two-fold branched cover of the link complement, and certain functions of it are true link invariants. For instance, the absolute value of its determinant is the absolute value of the determinant of the link. Further, G^{-1} is a matrix of the linking form on $H_1(D_L)$. I shall show later in this section that, up to the writhe and total linking number, Kauffman's two-variable polynomial $F_L(a, z)$ is a function of G. This raises the (unanswered) question of precisely what $F_L(a, z)$ has to do with, for example, this linking form.

The *graph* of a unoriented link diagram is constructed in a similar way. Take a vertex v_i in each black region R_i of the chess board coloured link diagram. Now for each crossing c at which R_i meets R_j, add an edge joining the corresponding vertices v_i, v_j. This edge is labelled with the sign $\xi(c)$ of the crossing. This construction provides us with a (signed) planar graph with a particular planar embedding. Conversely, given a planar embedding of a signed graph, one can construct a corresponding link diagram by placing a crossing of the appropriate sign in the middle of each edge and connecting these by arcs that run parallel to the edges of the graphs until they meet in neighbourhoods of the vertices. The graph is connected if and only if the link diagram is. See Figure 6 for an example and [2] for more details.

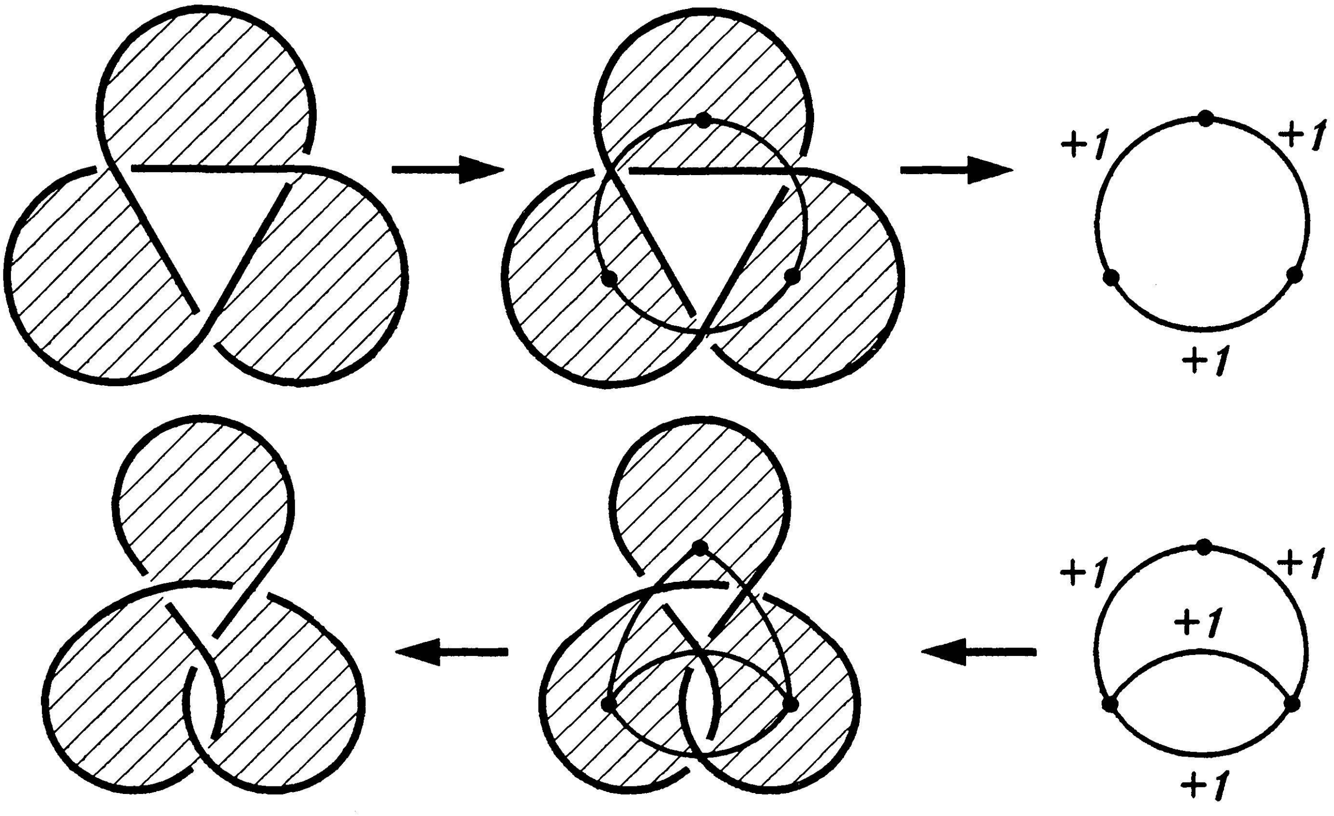

FIGURE 6

Notice that the graph of a connected diagram contains strictly more information than the Goeritz matrix, all information about the particular planar embedding and about loops in the graph being lost. Indeed, we can make use of this to construct diagrams of distinct links with identical Goeritz matrices, by picking graphs with more than one planar embedding. However, the variation that occurs here can be kept under tight control and I will make use of this fact later in this section.

2.2. KAUFFMAN'S POLYNOMIAL AND THE GOERITZ MATRIX

I now proceed to the main result of this section, linking the Goeritz matrix with Kauffman's F-polynomial invariant. Recall the observation made in section I that $\tilde{F}_L(a, z) = F_L(a, z) \cdot a^{\lambda/2}$ is invariant under change of orientation of components of L (where λ is defined to be the total linking number of L). Equivalently, we can define $\tilde{F}_L(a, z)$ by

$$\tilde{F}_L(a, z) = \Lambda_{\mathscr{D}}(a, z) \cdot a^{-w'} \tag{24}$$

where $\mathscr{D}$ is a diagram of the link L and w' is the *proper writhe* of $\mathscr{D}$, defined to be the algebraic sum of the signs of all crossings where a component of L meets itself. Note that the sign of such a crossing can be defined independently of any orientation on the link L (here I am speaking of crossing signs in the sense of section 1, not of the value ξ defined by a chess-board colouring of a diagram).

The following will be proved:

THEOREM 4. *The invariant* $\tilde{F}_L(a, z)$ *for a link* L *is a function of the Goeritz matrix of any diagram* $\mathscr{D}$ *of* L. □

Before proving Theorem 4, I must digress once again into graph theory. Recall that a graph is said to be *k-connected* if any $k - 1$ vertices (and their adjoining edges) may be removed without disconnecting the graph. The following result is due to Whitney ([27], [28]).

THEOREM 5. *Any planar embedding of a 3-connected graph is essentially unique.* □

The word "essentially" here means that we regard as equivalent any two embeddings which are ambient isotopic, any region of the graph's complement in the plane may be chosen to be the infinite region (this corresponds to a choice of region to contain the point at infinity in an embedding in the sphere) and the embedding may be reflected in some line in the plane. For more details, see [27], [28].

COROLLARY 6. *Let* P_1 *and* P_2 *be two planar embeddings of a connected graph* G. *Then there exists a finite sequence of the following moves which will transform* P_1 *into* P_2:

I. Ambient isotopy.

II. Reflection in a line.

III. The move illustrated in figure 7a).

IV. The move illustrated in figure 7b).

V. The move illustrated in figure 7c).

Proof. Proceed by induction on the number n of edges of G. Clearly the result is trivially true if $n = 0$. Now suppose it true far all connected graphs with $< n$ edges, and let G have n edges. If G is 3-connected then the result follows from Theorem 5. Otherwise there is a vertex or pair of vertices whose removal disconnects G. I consider these two cases separately.

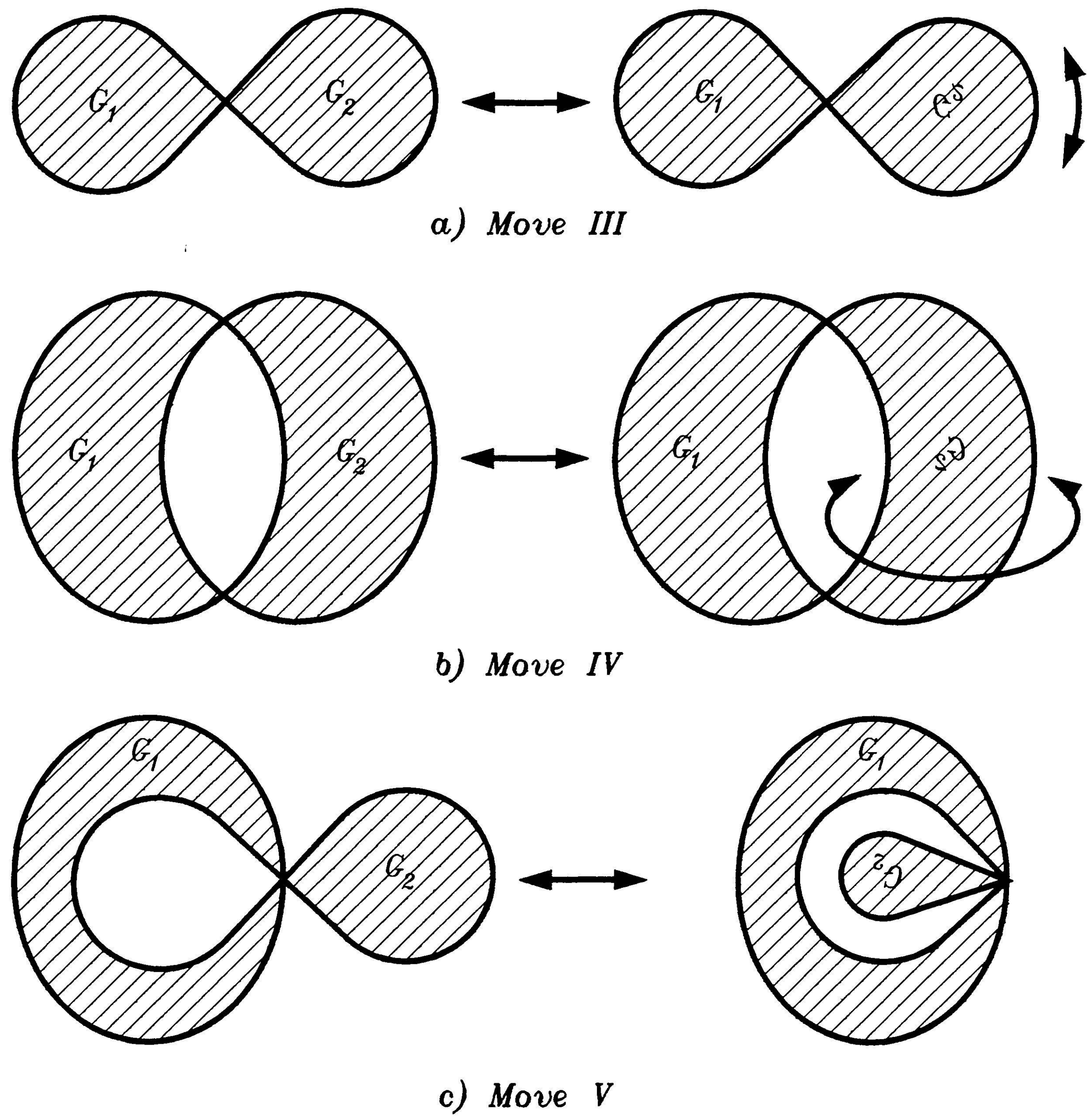

FIGURE 7

1) If there is a single vertex v whose deletion disconnects G, let G_1 and G_2 be the graphs obtained from the two components by adding to each a copy of v. Each of G_1, G_2 has fewer edges than G and so by the inductive hypothesis each satisfies the result. Now G is obtained from these two graphs by identifying the two copies of the vertex v and a planar embedding of G is specified by giving planar embeddings of G_1 and G_2 and specifying which planar region adjacent to the copy of v in each is occupied by the other. These different possibilities are all accounted for by moves III and V. Moves I, III, IV and V on G_1 and G_2 individually just correspond to the same moves on G, and move II on G_1 or G_2 corresponds to move III on G.

2) If there is a pair of vertices u and v whose removal separates G but no such single vertex, let G_1 and G_2 be the graphs obtained from the two components by adding to each copies of u and v. Each of G_1 and G_2 has fewer edges than G and so by the inductive hypothesis each satisfies the result. Now G is obtained from these two graphs by identifying the copies of u and v, and a planar embedding of G is specified by giving planar embeddings of G_1 and G_2 and specifying which planar region adjacent both to the copy of u and the copy of v in each is occupied by the other. These different possibilities are all accounted for by move IV. Moves I, III, IV and V on G_1 and G_2 correspond to the same moves on G, and move II on G_1 or G_2 corresponds to move IV on G. Hence the result is true of G and the induction proceeds. □

(In fact, move II is redundant since it follows from move IV, the subgraph on the left of the two chosen vertices in Figure 7b) consisting of a single edge joining those two vertices. Similarly, move III may be constructed from move IV, the subgraph to the left of the two chosen vertices in Figure 7b) being disconnected. I include these moves for clarity, however.)

Now consider the effects these moves on planar graphs have upon the corresponding link diagrams. Ambient isotopy of the graph merely corresponds to ambient isotopy of the link diagram. Reflection of the graph in some line corresponds to a reflection of the link diagram in that line followed by a reflection in the plane, the net effect of which is to rotate the link through 180 degrees about the line (see Figure (8)). Hence this does not change the link type corresponding to the graph's planar embedding.

Observe that if the signed graph G of a link diagram $\mathscr{D}$ has a cut-vertex v as in moves III and IV, then $\mathscr{D}$ is a connected sum. Figure 9 shows that move III corresponds to breaking up a connected sum and reconstituting it after reversing one of the summands. This may alter the link type of the connected sum (if at least one of the summands differs from its reverse), but does not affect the $\tilde{F}$-polynomial, since for any links L_1, L_2, we have

$$\tilde{F}_{L_1 \# L_2} = \tilde{F}_{L_1} \cdot \tilde{F}_{L_2}, \tag{25}$$

independent of the particular connected sum taken.

Similarly, move V corresponds to breaking up a connected sum and then reconstituting it, possibly summing together different components of the links. Again, this does not affect $\tilde{F}_L(a, z)$.

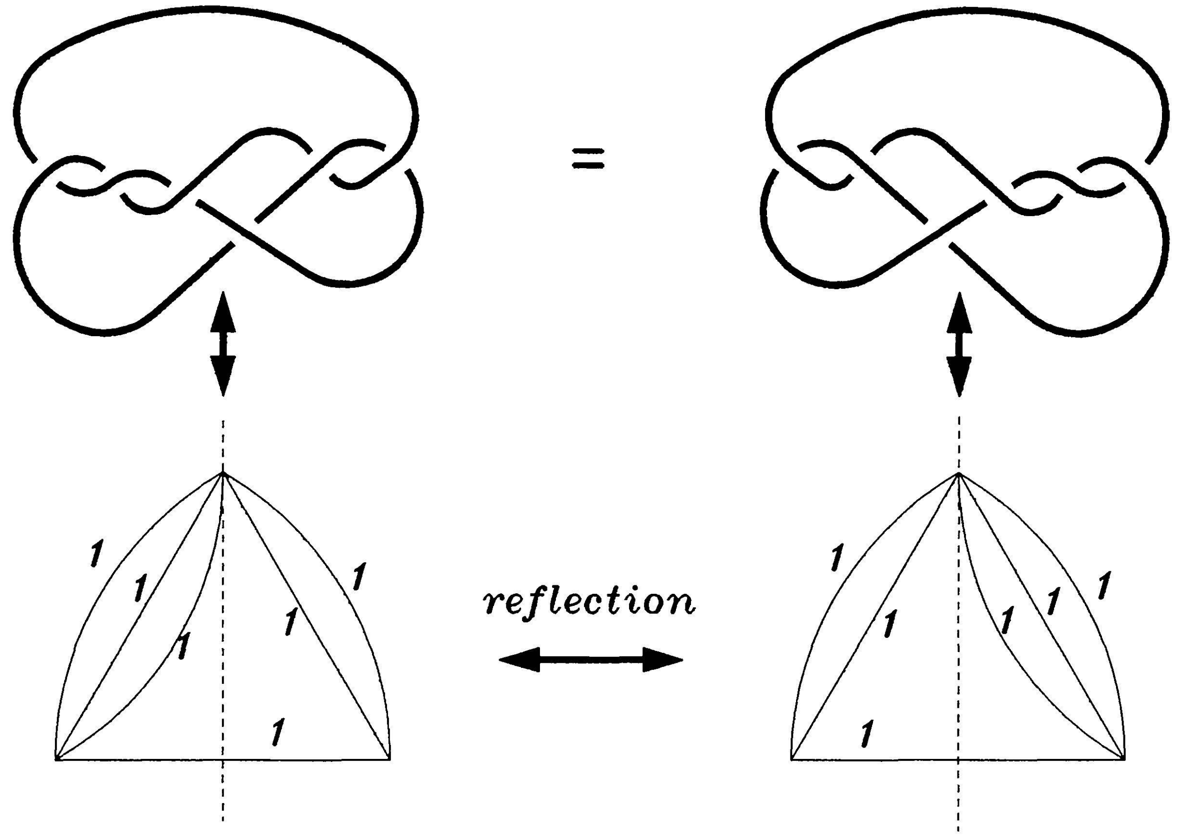

FIGURE 8

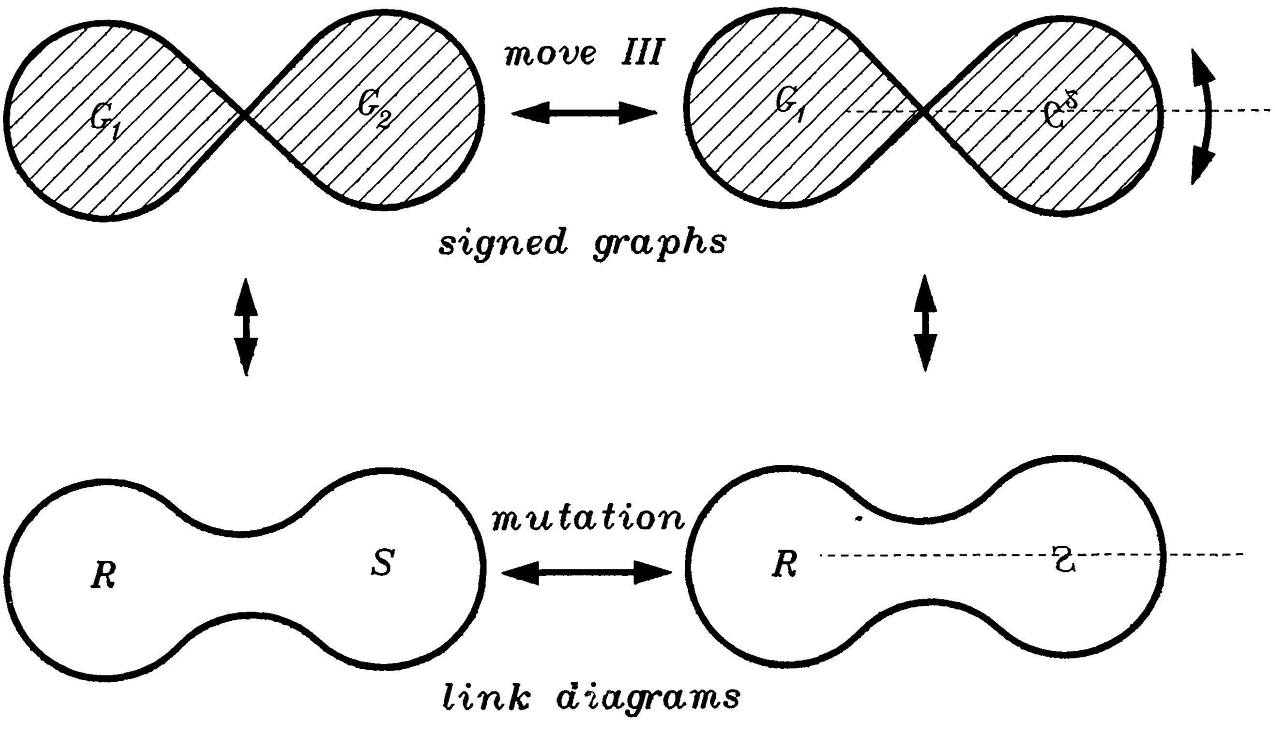

FIGURE 9

This leaves only move IV to be analysed. Figure 10 shows that this move corresponds to mutation of the underlying link. Once more, this leaves the $\tilde{F}$-polynomial unchanged.

The preceding discussion proves

THEOREM 7. *Given a link* L *with link diagram* $\mathscr{D}$, *the* $\tilde{F}$*-polynomial of* L *depends only on the isomorphism class of the signed graph corresponding to* $\mathscr{D}$, *and is independent of any particular planar embedding chosen.* □

In fact the same argument shows

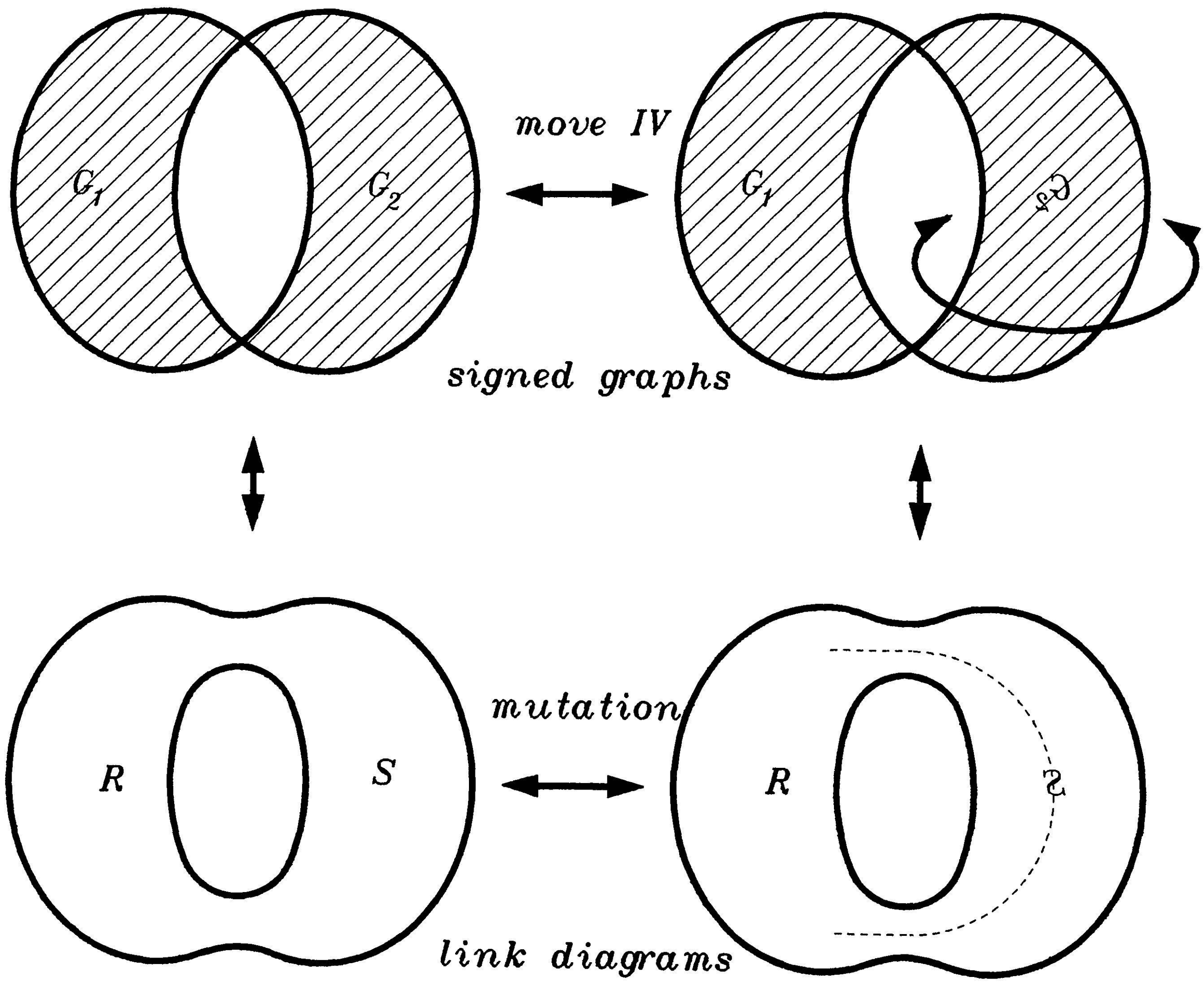

FIGURE 10

LEMMA 8. *Given a link diagram* $\mathscr{D}$, *the regular isotopy invariant* $\Lambda_{\mathscr{D}}(a, z)$ *depends only on the isomorphism class of the signed graph corresponding to* $\mathscr{D}$, *and is independent of any planar embedding information.* □

I can now proceed to the

Proof of Theorem 4. This follows from Theorem 7. The only information retained by the graph of a link diagram which is lost in passing to a Goeritz matrix is

(1) The number of edges of a given sign there are joining any particular pair of vertices. For each such pair the Goeritz matrix retains only the sum of the signs of these edges. But in terms of a chess-board colouring of the link diagram, this is to say that only the sum of the signs of crossings joining any two coloured regions R_i and R_j is retained. Suppose given a link diagram $\mathscr{D}$ with a chess-board colouring and two coloured regions R_i, R_j. Figure (11) shows that if R_i and R_j are connected by both a positive crossing and a negative crossing then by mutation of the link diagram these crossings can be made to cancel each other out.

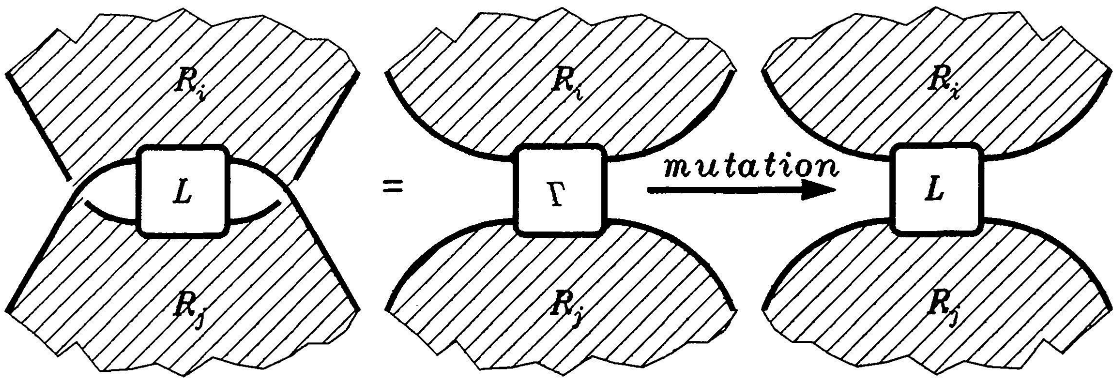

FIGURE 11

But mutation leaves $\tilde{F}_L(a, z)$ unaffected so it follows that only the sum of the signs of crossings joining each pair of coloured regions in $\mathscr{D}$ is relevant to calculation of $\tilde{F}_L$.

(2) The number of loops. However, loops in the graph correspond (possibly after an application of move V to the corresponding signed graph, see Figure 12) to Reidemeister-I style loops. These do not affect $F_L(a, z)$.

The theorem follows immediately. □

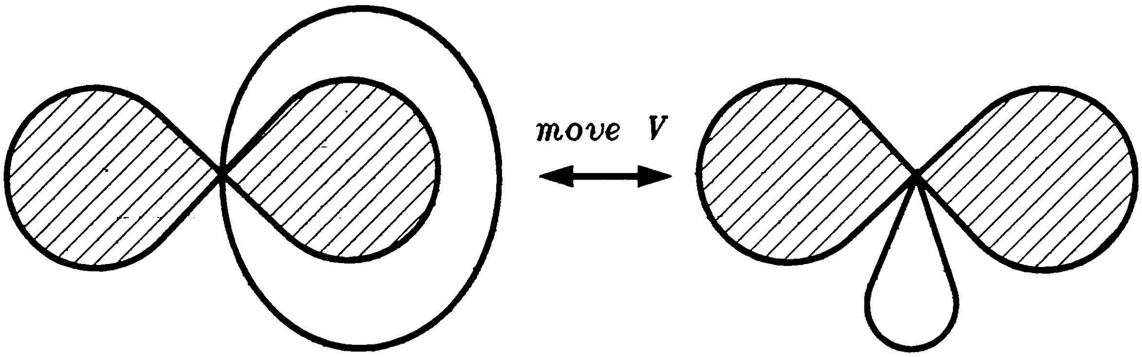

FIGURE 12

I conclude this section with an interesting observation. Gordon and Litherland [6] defined a signature σ_L for an unoriented link (differing from the classical signature of an oriented link by a term which is essentially the total linking number) and showed that it may be calculated from the signature of a Goeritz matrix by using a correction term calculated from the "types" of crossings in the associated diagram (Figure (13)).

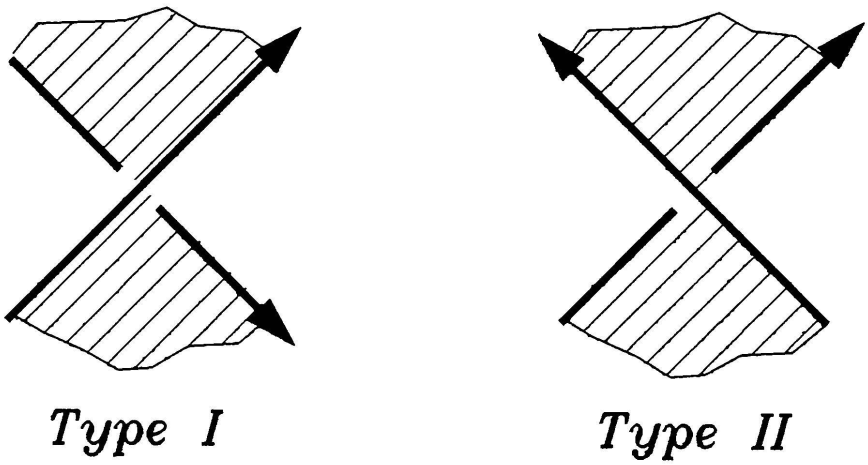

FIGURE 13

Their expression for σ_L is

$$\sigma(L) = \sigma(G) - \sum_{\mathrm{II}} \xi(c)\,, \tag{26}$$

the sum being taken over all crossings of type II where a single component of the link meets itself (at such crossings the type can be determined without ascribing an orientation to the link; for oriented links one sums over all crossings of type II to obtain the classical signature). But following through the same argument as above for the $\tilde{F}$-polynomial, we can show that the proper writhe of a "reduced" diagram (i.e. one with neither loops nor isthmuses in the corresponding signed graph) is a function of the Goeritz matrix (One uses precisely the same reasoning: Examine the effects of the moves of Corollary 6 and show that only the sums of signs of crossings joining adjacent regions are relevant). So given a Goeritz matrix for a diagram $\mathscr{D}$, the proper writhe of the diagram can be used to calculate the number of loops and isthmuses in the corresponding signed graph. Hence Gordon and Litherland's correction term $\sum_{\mathrm{II}} \xi(c)$ can be calculated from the Goeritz matrix and the proper writhe of the diagram, and conversely the proper writhe can be obtained from the Goeritz matrix and this term. So in the presence of the proper writhe of a diagram, the Goeritz matrix can be used to calculate the (unoriented link) signature σ_L.

Now, Thistlethwaite in [21], and Murasugi in [17] have proved

LEMMA 9. *The (proper) writhe of an alternating reduced diagram of a link L is an invariant of the link.* □

from which follows:

LEMMA 10. *The signature of an alternating unoriented link is a function of any Goeritz matrix for that link.* □

This should be compared with the result, also in [21] and [17]:

LEMMA 11. *The (classical) signature of an alternating link is a function of the F-polynomial of the link.* □

Theorem 4 raises the interesting question of what relation there is between $F_L(a, z)$ and the quadratic forms represented by Goeritz matrices. In particular, can either of the last two results be improved to cover non-alternating links?