

4. A LONG-STANDING CONJECTURE !

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

THEOREM 3.5 (S. A. Williams). *Let Ω be a bounded open subset such that the equation $\Delta T + \alpha^2 T = -1_\Omega$ has a function solution of compact support for some $\alpha > 0$. Let R, K, L be positive real numbers such that $L > KR$. Assume that for $P \in \partial^* \Omega$ there exists a coordinate system (x, y) around P so that*

- (i) $Q = (-R, R) \times (-L, L)$ intersects $\partial\Omega$ in the graph $y = f(x)$ of a Lipschitz function f with Lipschitz constant K , and
- (ii) $Q \cap \Omega = \{(x, y) : |x| < R \text{ and } f(x) < y < L\}$.

Then f is real analytic in a neighbourhood of P .

Thus if we restrict ourselves to the class \mathcal{D} of simply connected bounded open sets with Lipschitz boundary then $\Omega \in \mathcal{D}$ can fail to have the Pompeiu property only if $\partial\Omega$ is real analytic.

4. A LONG-STANDING CONJECTURE !

The following Conjecture has received quite some attention in the literature ([3], [10], [34]).

Conjecture. If $\Omega \subseteq \mathbf{R}^2$ is in the class \mathcal{D} described above and if Ω does not have the Pompeiu property, then Ω is a disc.

As pointed out before, the work of Williams shows that it is enough to consider Ω with $\partial\Omega$ real analytic. For $\Omega \in \mathcal{D}$, the existence of (a necessarily positive) α^2 for which (3.1) has a distribution solution of compact support is equivalent to the existence of a positive γ for which the following overdetermined system has a solution.

$$(4.1) \quad \begin{aligned} \Delta T + \gamma T &= 0 \quad \text{on } \Omega \\ T &= \text{constant} \neq 0 \quad \text{on } \partial\Omega, \quad \partial T / \partial n \equiv 0 \quad \text{on } \partial\Omega \end{aligned}$$

(see [34] for details). Thus the conjecture can be stated as follows:

If for $\Omega \in \mathcal{D}$, there exists $\gamma > 0$ for which (4.1) admits a solution, then Ω is a disc.

It is remarked in [34] that the conjecture is closely related to a result of Serrin ([25]): If Ω is a bounded connected open set with smooth boundary on which

$$\begin{aligned} \Delta u &= -1 \quad \text{on } \Omega \\ u &= 0, \quad \partial u / \partial n = \text{constant} \quad \text{on } \partial\Omega \end{aligned}$$

has a function solution, then Ω must be a disc.

We now state two partial answers to the conjecture that seem to support the conjecture.

THEOREM 4.1 (Berenstein [3]). *Let Ω be a simply connected bounded open subset of \mathbf{R}^2 with $C^{2+\varepsilon}$ boundary, where $\varepsilon > 0$. Assume that the boundary value problem (4.1) has solutions for infinitely many positive γ , then Ω is a disc.*

We need some notation for the next result due to Brown and Kahane ([10]). Let Ω be a convex bounded open connected subset of \mathbf{R}^2 . For $0 \leq \theta < \pi$, let $\omega(\theta)$ be the distance between the two parallel support lines for Ω which make an angle θ with the positive real axis. We assume $\partial\Omega$ is smooth so that ω is a continuous function. Let

$$m(\Omega) = \inf \{ \omega(\theta) : 0 \leq \theta < \pi \} \quad \text{and} \quad M(\Omega) = \sup \{ \omega(\theta) : 0 \leq \theta < \pi \}.$$

THEOREM 4.2 (Brown and Kahane [10]). *Let Ω be a convex region of \mathbf{R}^2 with $\partial\Omega$ real analytic. If $m(\Omega) \leq \frac{1}{2} M(\Omega)$, then Ω has the Pompeiu property.*

We remark that the proof of this Theorem is elementary and very elegant.

5. POMPEIU PROPERTY IN NON-COMPACT SYMMETRIC SPACES

Let G be a connected non-compact semisimple Lie group having finite centre and real rank 1. Let K be a fixed maximal compact subgroup of G . The space G/K is then a globally symmetric space of the non-compact type of rank 1. G/K is equipped with a natural Riemannian structure with respect to which G acts as a group of isometries and the associated Riemannian volume element μ is G -invariant. The basic results for the Pompeiu problem in this set-up are due to Berenstein and Zalcman ([9], [4]) and Berenstein and Shahshahani ([7]). In [9], the Fourier-analytic characterisation of a set — in fact, more generally, a collection of sets — having the pompeiu property is obtained and some explicit computations are made for geodesic spheres. In [7], the Pompeiu problem is reduced to an eigenvalue problem as in Section 4 and the analogue of Williams's results is obtained. We shall mainly present here a result implicit in the work of Berenstein and Zalcman as well as Berenstein and Shahshahani from our