II. PICARD-FUCHS EQUATIONS

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II. Picard-Fuchs equations

We will give a proof of Mordell's conjecture for function fields using Theorem 1.4.3 above. This theorem is weaker than Manin's Theorem of the Kernel (Theorem 2.1.0, below). In an appendix, we will give Chai's demonstration of Theorem 2.1.0 and show how Manin used it to complete his proof.

1. Picard-Fuchs differential equations

Let $f: X \to S$ be a smooth proper morphism with geometrically connected fibers over K. Let $\omega_{X/S} = H^0(X, \Omega^1_{X/S})$. Let Z be a subscheme of X finite over S whose normalization is smooth over S. Then $\omega_{X/S}$ injects naturally into both $H^1_{DR}(X/S)$ and $H^1_{DR}(X/S, Z)$ such that the obvious diagram commutes. Let W denote the image of $\omega =: \omega_{X/S}$ in $H^1_{DR}(X/S)$.

Let s and t be two sections of X/S, and $Z = s \cup t$. It follows that, if $s \neq t, H^1_{DR}(X/S, Z)$ is an extension of $H^1_{DR}(X/S)$ by $K[S]$ with a section on W. Hence we have an element $N(s, t)$ in $\mathrm{Ext}(H^1_{DR}(X/S), \mathscr{O}_S, W)$ which maps to $M(s, t)$ under the natural forgetful map from $\mathrm{Ext}(H^1_{DR}(X/S), \mathscr{O}_S, W)$ to $\mathrm{Ext}(H^1_{DR}(X/S), \mathscr{O}_S)$.

Now let $\mathscr{D} =: \mathscr{D}_S$ denote the algebra of differential operators on S, i.e. the free left algebra over $K[S]$ generated by $\mathrm{Der}_S =: \mathrm{Der}_{S/K}$. Since Der_S acts on the sections of a connection on S so does $\mathscr{D}$. Let $PF =: PF(X/S)$ denote the kernel of the natural map from $\mathscr{D} \otimes_{K[S]} \omega$ (where here $K[S]$ acts on $\mathscr{D}$ on the right) into $H^1_{DR}(X/S)$. Clearly, PF is a left $\mathscr{D}$-module. We call the elements of PF, Picard-Fuchs differential equations. The image of PF, under the natural map from $\mathscr{D} \otimes_{K[S]} \omega$ into $H^1_{DR}(X/S, Z)$, lies in the image of $K[S]$. We have the commutative diagram:

$$
\begin{array}{ccccc}
PF & \searrow & & & \\
 & & \mathscr{D} \otimes \omega & & \\
\downarrow & & \downarrow & \searrow & \\
K[S] & \to & H^1_{DR}(X/S, Z) & \to & H^1_{DR}(X/S)
\end{array}
$$

If $\mu \in PF$, call its image under the map to $K[S]$ $\mu(s, t)$. It follows from Proposition 1.3.1 that

$$\mu(r,s) + \mu(s,t) = \mu(r,t) \tag{1.1}$$

for $r, s, t \in X(S)$.

Suppose A/S is an Abelian scheme over S with origin section e. Then it follows from Theorem 1.4.1 that if $\mu \in PF(A/S)$, $s \to \mu(e,s)$ is a homomorphism from $A(S)$ into $K[S]$.

Manin's Theorem of the Kernel is:

THEOREM 2.1.0. *Suppose* $s \in A(S)$. *Then* $\mu(e,s) = 0$ *for all* $\mu \in PF(A/S)$ *iff* s *is a constant section.*

We will now explain the connection between this theorem and Theorem 1.4.3. Let w denote the natural map from $\mathrm{Ext}(H^1_{DR}(X/S), \mathscr{O}_S, W)$ to $\mathrm{Ext}([W], \mathscr{O}_S, W)$.

PROPOSITION 2.1.1. *Suppose* $s, t \in X(S)$. *Then* $\mu(s,t) = 0$ *for all* $\mu \in PF(X/S)$ *iff* $w \circ N(s,t) = 0$.

Proof. First, $[W]$ is the image of $\mathscr{D} \otimes \omega_{X/S}$ in $H^1_{DR}(X/S)$. Hence, if $\mu(s,t) = 0$ for all $\mu \in PF(X/S)$, we can define a horizontal section from $[W]$ to $E(s,t)$ by sending the image of an element of $\mathscr{D} \otimes \omega_{X/S}$ in $H^1_{DR}(X/S)$ to its image in $E(s,t)$. This implies $w \circ N(s,t) = 0$. The other direction is just as easy. □

Hence Manin's Theorem of the Kernel is equivalent to:

THEOREM 2.1.0′. *The class* $w \circ N(e,s) = 0$ *iff* s *is a constant section of* A/S.

On the other hand, it is easy to see that Theorem 1.4.3 is equivalent to this statement with $w \circ N(e,t)$ replaced by $N(e,t)$. Thus Theorem 2.1.0 follows from Theorem 1.4.3 in the case $[W] = H^1_{DR}(A/S)$, i.e.

PROPOSITION 2.1.2. *Suppose* $[W] = H^1_{DR}(A/S)$ *and* $s \in A(S)$. *Then* $\mu(e,s) = 0$ *for all* $\mu \in PF(A/S)$ *iff* s *is a constant section.*

Remark. The error in Manin's proof of Theorem 2.1.0 occurs in §6.2 on Page 214 of [*M*]. The displayed equation on line 12 is false. To make this statement true one must replace $\mathbf{r}$ with $\mathbf{r}^{\sigma}$, (in Manin's notation). In Appendix 1, we give Chai's proof that $N(e,t) = 0$ iff $w \circ N(e,t) = 0$ which together with Theorem 1.4.3 implies Theorem 2.1.0. However, we show below, that Proposition 2.1.2 is sufficient to prove the function field Mordell conjecture.

We call the composition

$$H^0(X, \Omega^1_{X/S}) \to H^1_{DR}(X/S) \overset{\nabla}{\to} \Omega^1_S \otimes H^1_{DR}(X/S) \to \Omega^1_S \otimes H^1(X, \mathscr{O}_X),$$

where the maps on either end are natural ones, the Kodaira-Spencer map and denote it by $\kappa_{X/S}$. An important special case of the previous proposition is the one in which $\kappa_{X/S}$ is an isomorphism, since then

$$(\Omega^1_S \otimes W) \oplus \kappa_{X/S} W \cong \Omega^1_S \otimes H^1_{DR}(X/S)$$

under the natural map and so, in particular, $[W] = H^1_{DR}(X/S)$. It is well known that if X is a family of curves over S and the Kodaira-Spencer map is zero then X/S is an isoconstant family, i.e., becomes constant after a finite base extension.

PROPOSITION 2.1.3. *Suppose* $\mathrm{Der}_{S/K}$ *is spanned by* ∂ *over* $K[S]$. *Suppose* $\kappa_{X/S}$ *is an isomorphism. There exists a* $K[S]$*-linear map from* $\omega_{X/S}$ *to* PF

$$\omega \in \omega_{X/S} \to \mu_{\partial,\omega} =: \mu_\omega,$$

characterized by the condition that μ_ω *can be written in the form* $\partial^2 \otimes \omega + \partial \otimes \omega' + 1 \otimes \omega''$, *where* ω' *and* $\omega'' \in \omega_{X/S}$. *Moreover* PF *is generated over* $\mathscr{D}$ *by the image of this map.*

Proof. The fact that $(\Omega^1_S \otimes W) \oplus \kappa_{X/S} W \cong \Omega^1_S \otimes H^1_{DR}(X/S)$ implies that there exist unique elements ω' and ω'' in W such that $\partial^2 \otimes \omega + \partial \otimes \omega' + 1 \otimes \omega'' \in PF$. The $K[S]$-linearity follows from the uniqueness and fact that for any $\nu \in \omega_{X/S}$, $n \in \mathbf{Z}_{\geqslant 0}$ and $f \in K[S]$, one may write $f\partial^n \otimes \nu$ in the form $\partial^n \otimes f\nu + \sum_{0 \leqslant i < n} \partial^i \otimes \nu_i$ with $\nu_i \in \omega_{X/S}$. The fact that PF is generated by these elements is also clear. □

COROLLARY 2.1.4. *Suppose* $\mathrm{Der}_{S/K}$ *is spanned by* ∂ *over* $K[S]$. *Suppose* $\kappa_{A/S}$ *is an isomorphism. Then*

$$\{s \in A(S) : \mu_{\partial,\omega}(e, s) = 0\} = A(S)_{\mathrm{tor}}.$$

Proof. This follows immediately from Theorem 2.1.2 since the only constant sections in this case are torsion.

2. Picard-Fuchs computations

We will need an explicit formula for $\mu(s, t)$ in some cases. Suppose that X/S has relative dimension one. Suppose $z \in K[S]$ such that $\Omega^1_S(S) = K[S]dz$ and suppose U is an affine open of $X, s \in U(S)$ and $v \in \mathscr{O}_X(U)$, such that $s^*v = 0$ and $\Omega^1_{X/S}(U) = \mathscr{O}_X(U)d_{X/S}v$. For $u \in \mathscr{O}_X(U)$ we define $\partial_z u$ and $\partial_v u$ by

$$du = \partial_z u dz + \partial_v u dv$$

Clearly ∂_z is a lifting of $\partial =: \partial/\partial z$ to a derivation of $\mathscr{O}_X(U)$. For $\omega = ud_{X/S}v \in \Omega^1_{X/S}(U)$ we set $\partial_z\omega = \partial_z u d_{X/S}v$ (the image of the Lie derivative of udv with respect to ∂_z in $\Omega^1_{X/S}(U)$). Since ∂ generates $\mathscr{D}$ over $K[S]$ we can and will also make $\mathscr{D}$ act on $\Omega^1_{X/S}(U)$ using ∂_z.

Lemma 2.2.1. *Suppose* $\omega = ud_{X/S}v \in \Omega^1_{X/S}(U)$ *is of the second kind and* $[\omega]$ *is its class in* $H^1_{DR}(X/S)$. *Then*

$$\partial[\omega] = [\partial_z\omega] .$$

Proof. The element udv is a lifting of $ud_{X/S}v$ to $\Omega^1_X(U)$, and $d(udv) = du \wedge dv = \partial_z udz \wedge dv$. Since this is the image of $dz \otimes \partial_z\omega$ in Ω^2_X the lemma follows. □

Corollary 2.2.2. *Suppose* $\sum D_i \otimes \omega_i \in PF$. *Then*

$$\sum D_i\omega_i = d_{X/S}w$$

for some $w \in \mathscr{O}_X(U)$.

Suppose $t \neq s$ is an element of $U(S)$ and $Z = s \cup t$. Let l denote the map from $K[S]$ into $H^1_{DR}(U/S, Z)$ associated to the pair (s, t). For $\omega \in \Omega^1_{X/S}(U)$ let $[\omega]_Z$ denote the class of ω in $H^1_{DR}(U/S, Z)$.

Lemma 2.2.3. *Suppose* U, s *and* v *are as above,* $t \in U(S)$ *and* $t^*v \neq 0$. *Suppose* $\omega = ud_{X/S}v \in \Omega^1_{X/S}(U)$. *Then* $\partial^k[\omega]_Z$ *equals*

$$[\partial^k_z\omega]_Z + l(\sum \partial^{i-1}(t^*(\partial^{k-i}_z u)\partial t^*v))$$

where i *runs from* 1 *to* k.

Proof. By shrinking S we may suppose that t^*v is invertible. We want to compute $\nabla[\omega]_Z$. First we must lift $ud_{X/S}v$ to section of $\Omega^1_{X,Z}(U)$. Let $y = f^*(t^*v)$. Then $\eta = uydy^{-1}v$ is such a lifting and it equals $udv - uvy^{-1}\partial_z ydz$. Then $\nabla[\omega]_Z$ is the class of

$$d\eta = \partial_z u dz \wedge dv - d(uvy^{-1}) \wedge dy = dz \wedge \partial_z u dv + dz \wedge d(uvy^{-1}\partial_z y) .$$

which is the image of

$$dz \otimes (\partial_z \omega + d_{X/S}(uvy^{-1}\partial_z y)) \in \Omega^1_S \otimes \Omega^1_{X/S}(U) .$$

Hence $\partial[\omega]$ is the class of $\partial_z\omega + d_{X/S}(uvy^{-1}\partial_z y)$ in $H^1_{DR}(U/S, Z)$. Since $(t^* - s^*)(uvy^{-1}\partial_z y) = t^*u\partial(t^*v)$ the lemma follows in the case $k = 1$. Since $\partial \circ l = l \circ \partial$ the lemma follows in general by induction. □

COROLLARY 2.2.4. *Suppose* U, s, z *and* v *are as above,* $t \in X(S)$ *which meets* U *and* $t^*v \neq 0$. *Suppose* ω, ω' *and* ω'' *are elements* $\omega_{X/S}$. Let $\omega = ud_{X/S}v$ *and* $\omega' = u'd_{X/S}v$ *on* U. *Then we have:*

(i) *Suppose* $\mu = \partial \otimes \omega - 1 \otimes \omega' \in PF$, $\omega = ud_{X/S}v$ *and* $\partial_z\omega - \omega' = d_{X/S}w$, *with* $w \in \mathscr{O}_X(U)$. *Then*

$$\mu(s, t) = t^*w - s^*w + (t^*u)\partial t^*v .$$

(ii) *Suppose* $\mu = \partial^2 \otimes \omega + \partial \otimes \omega' + 1 \otimes \omega'' \in PF$ *and* $\partial^2\omega + \partial\omega' + \omega'' = d_{X/S}w$ *with* $w \in \mathscr{O}_X(U)$. *Then*

$$\mu(s, t) = t^*((w - s^*w, (u' + 2\partial_z u), \partial_v u, u) \cdot (1, x_t, x_t^2, \partial x_t))$$

and where $x_t = \partial t^* v$.

Proof. First shrink S so that s and t satisfy the hypotheses of the lemma and then apply it and the definition of $\mu(s, t)$. □

Suppose $g: X \to A$ is a morphism over S from a curve to an Abelian scheme. Suppose $\kappa_{A/S}$ is an isomorphism. If $\eta = g^*\omega$ where $\omega \in \omega_{A/S}$ we will set $\mu_\eta = g^*\mu_\omega$. This is independent of the choice of ω. As an immediate consequence of the previous corollary we obtain:

COROLLARY 2.2.5. *Let* U, z, s *and* v *be as above. Set* $X(S)' = \{t \in X(S): t$ *meets* U *and* $t^*v \neq 0\}$. *Then there exist maps*

$$V =: V_{z,v}: T_{U,v} \to K(S)^4$$

and

$$L =: L_{z,v,s}: \omega_{X/S} \to K(X)^4$$

such that L *is K-linear and for* $t \in X(S)'$ *and* $\omega \in g^*\omega_{A/S}$,

$$\mu_\omega(s, t) = t^*(L(\omega) \cdot V(t)) .$$