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The models given in this paper do not share this ease of definition. Their well-definedness is just as easy or difficult as the standard inductive proof of the existence of the polynomial in question. To discriminate them from the Yang-Baxter and other models, I shall call them *skein models.* The skein models are, in fact, seen as translations (into the language of state models) of the recursive process of calculation due to John H. Conway ([16]). The term *skein* is due to Conway. It refers to all the knots and links associated with a given link that are obtained via splicing or switching some of its crossings. Skein calculation uses the knots and links in this skein.

The first skein model was discovered by François Jaeger [33], via a matrix inversion technique. In this paper I show that Jaeger's idea fits into the more general scheme of skein calculation, and hence applies fully to both of the known two-variable skein polynomials, and to their graph embedding generalizations ([56], [74]).

As explained herein, the skein models appear as tautologous — particularly to anyone who has written a computer program to calculate knot polynomials. This is the virtue of our approach! What is significant is that we have, in fact, been in possession of general states models for these polynomials for years. It took Jaeger's observation about standardized basepoints (called here a *template* — see section 2) to show what we already knew.

It is useful to know that general states models exist. And it is very fascinating to compare the form of the general models to the forms of specific models previously known (bracket, FKT, Yang-Baxter). See sections 7, 8, 9 of this paper, particularly section 9 for a discussion of possible physical interpretations. The appendix is a discussion of state model formalism.

## II. Skein polynomials

This paper concentrates on two polynomials — the *Homfly* ([24]) or *oriented skein polynomial* and the *Kauffman* or *semi-oriented skein polynomial* ([43], [49]). In both cases it is convenient to first define a polynomial that is an invariant of regular isotopy ([49]), and then normalize the regular isotopy invariant to obtain the corresponding skein polynomial.

NOTATION. *Regular isotopy* is the equivalence relation generated by Reidemeister moves II and III (see Figure 1). Regular isotopy of links $K$ and $L$ is denoted $K \approx L$. *Ambient isotopy* is the equivalence relation generated by all three Reidemeister moves, and will be denoted by $K \sim L$. (See [80] or [14] for a proof that this diagrammatic version of ambient isotopy agrees with the usual definition for embeddings of links in three dimensional Euclidean space.)

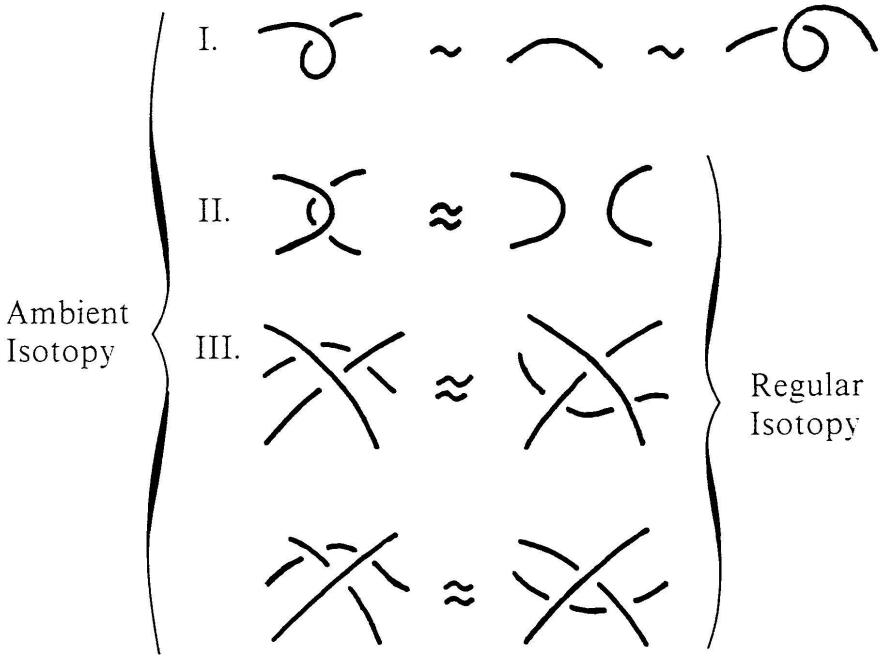



FIGURE 1
*The Reidemeister Moves*

In an oriented link, each crossing has a *sign* $\varepsilon = +1$ or $-1$, according to the convention shown in Figure 2.


FIGURE 2
*Crossing Signs*

The *writhe*, $w(K)$, of an oriented diagram $K$ is the sum of the signs of the crossings in the diagram. The writhe is an invariant of regular isotopy.

The *oriented skein polynomial*, $P_K(z, a)$, is described by two axioms as an ambient isotopy invariant:

1. $P_{\bigcirc} = 1$.

2. $aP_{\nearrow} - a^{-1}P_{\searrow} = zP_{\rightarrow}$.

3. $P_K(z,a) = P_{K'}(z,a)$ in $Z[z, z^{-1}, a, a^{-1}]$ whenever $K$ and $K'$ are ambient isotopic diagrams.

The three small diagrams in axiom 2 stand for parts of larger diagrams that differ only as shown in the small diagrams.

I shall use the following formulation for $P$:

$$P_K(z, a) = a^{-w(K)}R_K(z, a)$$

where $w(K)$ is the writhe of $K$, and $R_K(z, a)$ is a regular isotopy invariant satisfying the axioms:

1'. $R_{\bigcirc} = 1$,

$R_{\curvearrowright} = aR_{\rightarrow}$,

$R_{\curvearrowleft} = a^{-1}R_{\rightarrow}$.

2'. $R_{\nearrow} - R_{\searrow} = zR_{\rightarrow}$.

3'. $K \approx K' \Rightarrow R_K = R_{K'}$.

The identity of axiom 2'. will be referred to as the *Conway identity*. (If $a = 1$ then $R_K(z, 1) = \nabla_K(z)$ is the Conway-Alexander polynomial [16].)

It is known that these axiom systems are consistent, and that they can be used to recursively calculate the polynomials.

I shall also discuss the "Dubrovnik" version of the Kauffman polynomial [49]. This regular isotopy invariant, denoted $D_K(z, a)$, is defined for unoriented links $K$ via the axioms:

1". $D_{\bigcirc} = 1$, $D_{\curvearrowright} = aD_{\frown}$, $D_{\curvearrowleft} = a^{-1}D_{\frown}$.

2". $D_{\times} - D_{\times} = z(D_{\asymp} - D_{)(})$.

3". $K \approx K' \Rightarrow D_K = D_{K'}$.

The key is to see that an unoriented crossing can distinguish two pairs of local regions (designated $A$ and $B$) as shown in Figure 3.

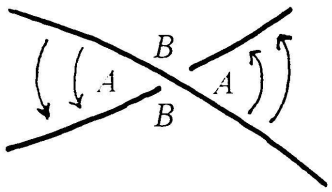



FIGURE 3

The $A$ regions are swept out first as the overcrossing line is turned counterclockwise.

With this in mind, axiom 2. makes sense by associating with each crossing on the left-hand side of the equation the corresponding splice on the right-hand side that joins the $A$-regions.

The ambient isotopy invariant, $Z_K$, for the Kauffman (Dubrovnik version) polynomial is given by the formula

$$Z_K(z, a) = a^{-w(K)} D_K(z, a) .$$

where $K$ is an oriented link ($D$ forgets the orientation).

These polynomials can be calculated systematically and recursively by using the concept of the *standard unlink* ([8]):

Given a universe (locally 4-valent plane graph), choose an edge and a direction on that edge. Traverse the universe, *crossing over* at the first passage through each crossing. (Upon returning to a crossing, cross under — so leaving the choice at first passage unchanged.) If this crossing circuit does not use up every crossing in the universe, then choose an edge in the complement of circuits traversed so far, and repeat the process.

The final result is an unlink (each component is unknotted and any two components are unlinked). See Figure 4.


*U*
*universe* *standard unlink*

FIGURE 4

The exchange identity (for either polynomial) expresses the value for a given link in terms of the value for a link obtained by switching one crossing, and the value of the polynomial on a link (or links) with fewer crossings. If the switched link is "more unlinked" than the given link, then this procedure proceeds inductively to calculate the polynomial.

The value of the standard unlink is that it can be used as a basis of comparison for the switching process. Having chosen a standard unlink, $K$, with the same underlying universe as $K$, we can decide to switch a crossing in $K$ exactly when it *differs* from the corresponding crossing on the standard unlink. Successive switchings then lead toward the standard unlink.

Of course, the calculation process also involves links obtained by splicing crossings of $K$. For these it is necessary to also choose standard unlinks.

Each standard unlink for $K$ is determined by a choice of base-point and direction of travel along $K$. Choosing a basepoint is equivalent to choosing an edge from the universe $U$ for $K$ (an edge extends from one crossing to another). We can choose all the basepoints for the whole calculation at once by making a *template*:

*Definition 2.1.* Let $K$ be a link diagram with underlying universe $U$. Suppose that $U$ has $n$ edges. A *template for* $K$ is a labelling of the edges of $U$ from the set $\{1, 2, ..., n\}$ when $K$ and $U$ are oriented. If $K$ is unoriented, then a template consists in a labelling of the edges of $U$, together with an orientation of each edge of $U$. (Note that there are no compatibility restrictions on these orientations, as in the orienting of a link diagram where travel in the direction of the orientation carries one through a crossing.)

A diagram obtained from $K$ (above) by splicing some crossings can be regarded as having extra 2-valent vertices at the sites of the splice. In this way *the template for* $K$ *is regarded as a template for all diagrams obtained from* $K$ *by either switching or splicing crossings.*

Let $K$ be a given link diagram and $T$ a template for $K$. The link components of $K$ correspond to circuits in the underlying universe $U$. (Such circuits cross at a crossing, and are characterized by this property.) Let $L$ be a link component of $K$.

*Definition 2.2.* The *index of* $L$ *with respect to the template* $T$, $I(L, T)$, is the least index in $T$ on the circuit corresponding to $L$. Let $\mathrm{Ind}(K)$ denote the collection of indices of components of $K$, arranged in increasing order.

With this notion of index we can specify the algorithm to compute either polynomial:

*Template Skein Algorithm*

0. This algorithm produces a tree rooted at the given link diagram $K$, with a collection of unlinks at its ends (farthest branches). The nodes of this tree are link diagrams.

1. Begin by letting $i$ denote the least index in Ind $(K)$.

2. Using the basepoint specified by the index $i$, find the first crossing that is an undercrossing (in the direction of travel from the basepoint). Call this crossing the *active crossing.*

3. Produce the switched and spliced diagrams associated to the active crossing (two diagrams in the oriented case, three in the semi-oriented case). *Regard the diagrams so produced as nodes on the tree, connected by edges of the tree with the source diagram* $K$.

4. If all crossings are inactive from a given basepoint, choose the next basepoint, $j$, in the order specified by Ind $(K)$. Using the basepoint specified by the index $j$, find the first crossing *of components of index at least equal to* $j$ that is an undercrossing (in the direction of travel from the basepoint). Produce new nodes (see 3. above) from this crossing.

5. If all crossings are inactive for all basepoints, then $K$ is a standard unlink. Call this an *end-node* in the tree.

6. Apply this process to each produced node until the tree is complete.

The value of the polynomial on any unlink is given by the formulas:

$$R_K = a^{w(K)}\delta^{|K|-1}$$
$$D_K = a^{w(K)}\mu^{|K|-1}$$
($K$ unlinked, $|K|$ = number of components of $K$)
$$\delta = (a-a^{-1})/z$$
$$\mu = 1 + (a-a^{-1})/z\,.$$

(This follows directly from the axioms.)

Call this tree the *template skein for* $K$ *relative to* $T$. See Figure 5.

*Definition 2.3.* Let $U$ be a universe with template $T$. A link $L$ is said to be a *standard unlink relative to* $T$ if the tree produced by the template skein algorithm for $L$ relative to $T$ consists in a single node.

Letting $E_T(K)$ denote the set of end-nodes (standard unlinks) produced by the template skein algorithm for $K$ relative to $T$, we have that

$$R_K = \sum (-1)^{t-(L)} z^{t(L)} a^{w(L)} \delta^{|L|-1}$$

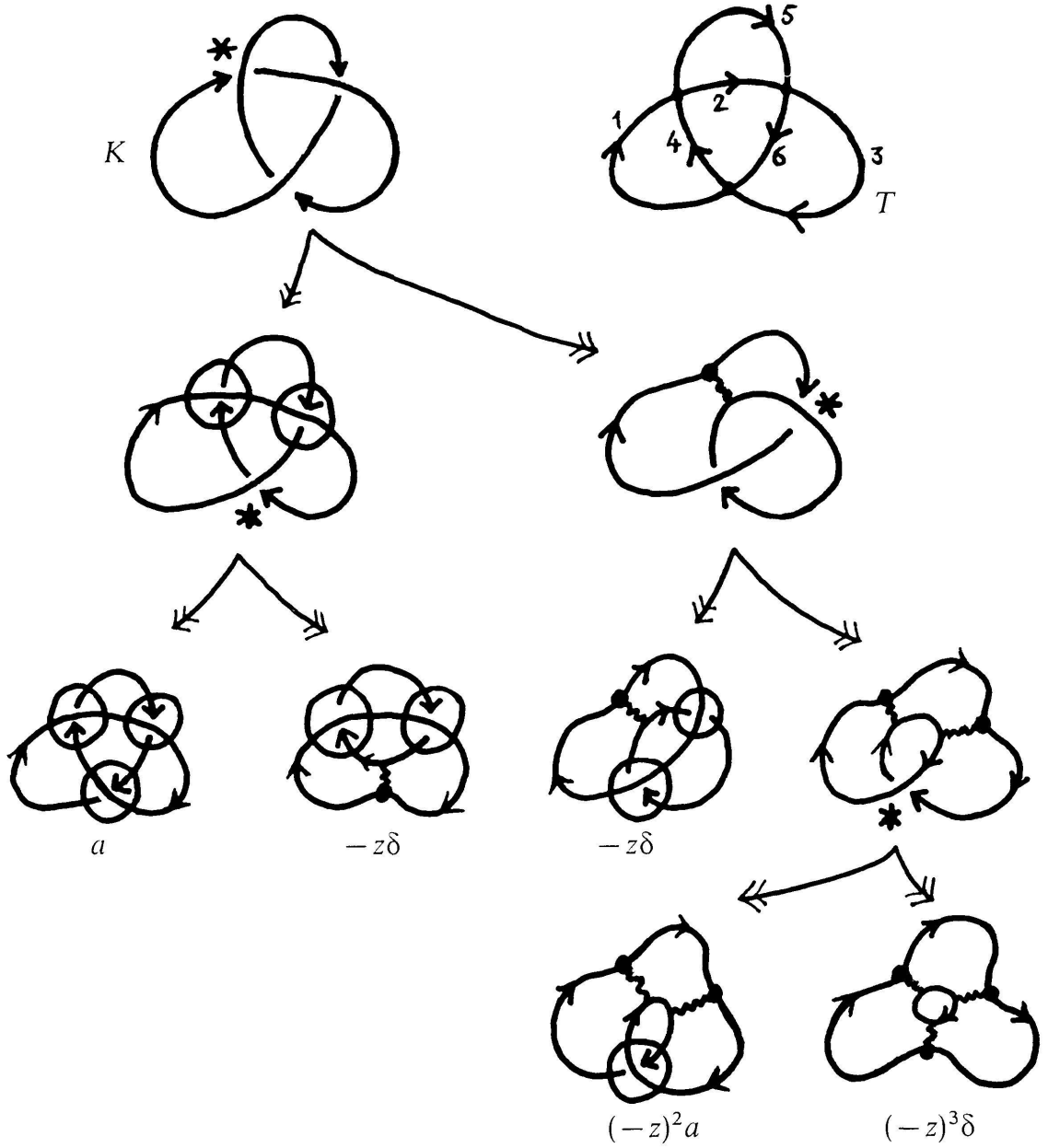
summation over $L \in E_T(K)$ ($E$ for oriented case, $E$ for unoriented case) and

$$D_K = \sum (-1)^{t-(L)} z^{t(L)} a^{w(L)} \mu^{|L|-1}$$

summation over $L \in E_T(K)$, where $t(L)$ denotes the number of splices

performed to obtain $L$ from $K$, and $t_-(K)$ is the number of splices from a crossing of negative sign in the Homfly case. In the Dubrovnik case, $t_-(K)$ denotes the number of splices that are of "$B$-type" (see Figure 3). This formula follows directly from the exchange identity, and the form of calculation.

The next two sections will characterize and reformulate the elements of $E(K)$, and thus construct the skein models.




$$E_T(K) = \{\ ,\ ,\ ,\ ,\ \}$$

$$R_K = a - z\delta - z\delta + z^2a - z^3\delta = a^{-1}z^2 + (za^{-1} - a)$$

FIGURE 5

*Template Skein Algorithm*