It will be sufficient to look only at the symmetries of H which take the fibre $L_0 = \{(u, 0)\}$ to itself, and hence are of the form $(u, v) \mapsto (A(u), B(v))$. We already know that there must be a $C \in SO(8)$ such that $B(mu) = C(m)\, A(u)$ for all $m, u \in Ca$. To show that G is a nontrivial double covering of $SO(9)$, we must find a *loop* of C's which lifts to a *non-loop* of (A, B)'s.

This can be done by using the Moufang identities, just as in the proof of the Triality Principle. Recall from that proof that if x is an imaginary Cayley number of unit length, then $A = L_x$, $B = -L_x$ and $C = L_x R_x$ "works", that is, $-L_x(mu) = L_x R_x(m)\, L_x(u)$. Now let x describe a semi-circular path in the i, j-plane from i to $-i$. At the beginning of the path, $C(m) = imi$, while at the end of the path $C(m) = (-i)m(-i) = imi$. Thus C describes a loop in $SO(8)$. At the beginning of the path, $(A(u), B(v)) = (iu, -iv)$, while at the end $(A(u), B(v)) = (-iu, iv)$. Hence (A, B) describes a non-loop in G. Thus G is the non-trivial double covering Spin(9) of $SO(9)$.

QED

Here is a further indication of the extent of symmetry of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$. Orient the fibres.

PROPOSITION 7.10. *Let P and Q be any two fibres of H. Then a preassigned orientation preserving rigid motion of P onto Q can be extended to a symmetry of H. In particular, the symmetries act transitively on S^{15}.*

By Lemma 7.7, the symmetries act transitively on fibres, so we may take $P = Q = L_0$. To preassign an orientation preserving rigid motion of L_0 onto itself is to preassign the map $A \in SO(8)$ in the Triality Principle, which then promises the desired symmetry of H. QED

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Herman Gluck
Frank Warner
Wolfgang Ziller

Department of Mathematics
University of Pennsylvania
Philadelphia, 19104-6395 (USA)