5. Normed Division Algebras and the Cayley Numbers

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isometry of the base space HP^{n-1} in its canonical metric. It is easy to check that when $n = 2$, every orientation preserving isometry of the base $HP^1 = S^4(1/2)$ can be produced this way, while no orientation reversing one can (since the group is connected). We remark without proof that *all* isometries of HP^{n-1}, $n > 2$, can be produced this way, and that they are all orientation preserving.

5. Normed Division Algebras and the Cayley Numbers

In order to describe the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$ in the next section, we first review here some facts about normed division algebras and the arithmetic of Cayley numbers. More can be found in two excellent references, [Cu] and [H-L, pp. 140-145].

A *normed division algebra* B is a finite dimensional algebra over the reals R, with multiplicative unit 1, and equipped with an inner product $< , >$ whose associated norm $| \ |$ satisfies

$$|xy| = |x|\,|y| \quad \text{for all} \quad x, y \in B\,.$$

By Hurwitz' Theorem ([Hu 1], 1898), a proof of which we will outline here, every normed division algebra is isomorphic to either the reals R, the complex numbers C, the quaternions H or the Cayley numbers Ca. Actually, what Hurwitz proved is that normed division algebras can only occur in dimensions 1, 2, 4 and 8. He stated the corresponding uniqueness result without proof. In [Hu 2], published in 1923 after his death, Hurwitz credits E. Robert [Ro] with writing out the details of the uniqueness argument in a 1912 Zurich thesis.

Now let B denote a given normed division algebra. Let $\operatorname{Re} B$ denote the one-dimensional linear subspace spanned by the identity 1, and $\operatorname{Im} B$ the orthogonal complement of $\operatorname{Re} B$. Then each $x \in B$ has a unique orthogonal decomposition,

$$x = x_1 + x'\,, \quad x_1 \in \operatorname{Re} B \quad \text{and} \quad x' \in \operatorname{Im} B\,,$$

into its real and imaginary parts. *Conjugation* in B is defined by:

$$\bar{x} = x_1 - x'\,.$$

Here are some basic facts about arithmetic in any normed division algebra B:

1) $<xw, yw> = <x, y> |w|^2 = <wx, wy>$.
Thus right or left multiplication by a unit vector w is an isometry of B.

2) Every nonzero $x \in B$ has a unique left and right inverse:

$$x^{-1} = \bar{x}/|x|^2 .$$

3) Given x and y in B with $x \neq 0$, the equations

$$xw = y \quad \text{and} \quad wx = y$$

can each be solved uniquely, with

$$w = x^{-1}y \quad \text{and} \quad w = yx^{-1}$$

respectively.

4) $$\overline{xy} = \bar{y}\,\bar{x} .$$

5) If x is imaginary (that is, $x \in \operatorname{Im} B$), then $x^2 = -|x|^2$.

6) Orthogonal imaginaries anti-commute. That is,

$$x, y \in \operatorname{Im} B \quad \text{and} \quad <x, y> = 0 \quad \text{imply} \quad xy = -yx .$$

7) The Moufang identities, the first two of which say that left and right multiplication by xyx can be performed successively:

$$(xyx)z = x(y(xz))$$
$$z(xyx) = ((zx)y)x$$
$$x(yz)x = (xy)(zx) .$$

Given three elements $x, y, z \in B$, their *associator* is defined by

$$[x, y, z] = (xy)z - x(yz) .$$

The following weak form of associativity always holds in a normed division algebra: the trilinear form $[x, y, z]$ is *alternating*, i.e., it vanishes whenever two of its arguments are equal. Such an algebra is said to be *alternative.*

The Cayley-Dickson process generalizes the way in which the complex numbers are built up from the reals, and begins with the following

PROPOSITION 5.1. (see [Cu] or [H-L]). *Let A be a subalgebra (containing 1) of the normed division algebra B. Let ε be an element of B orthogonal to A with $|\varepsilon| = 1$. Then*

i) *$A\varepsilon$ is orthogonal to A, and*

ii) $(a+b\varepsilon)(c+d\varepsilon) = (ac-\bar{d}b) + (da+b\bar{c})\varepsilon$ *for all a, b, c, d in A.*

The proof makes use of the commutation rules given in Fact 6 above.

We note for future use that the above proposition implies that any subalgebra of Ca generated by two elements must be isomorphic to R, C or H, and hence must be associative.

Suppose now that we start with a normed division algebra A and define a product on $A \oplus A$ by

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}).$$

The new algebra $B = A \oplus A$ is said to be obtained from A via the *Cayley-Dickson process*. In particular,

$$C = R \oplus R, \quad H = C \oplus C, \quad Ca = H \oplus H$$

via the Cayley-Dickson process.

PROPOSITION 5.2. (Jacobson [Ja], 1958). *Suppose $B = A \oplus A$ is obtained from A by the Cayley-Dickson process. Then*

1) *B is commutative $\Leftrightarrow$ $A = R$.*
2) *B is associative $\Leftrightarrow$ A is commutative.*
3) *B is alternative $\Leftrightarrow$ A is associative.*

See [Cu] or [H-L] for details.

From this proposition, we have:

$C = R \oplus R$ is commutative;

$H = C \oplus C$ is associative, but not commutative;

$Ca = H \oplus H$ is alternative, but not associative;

$Ca \oplus Ca$ is not alternative, *hence not a normed division algebra.*

THEOREM 5.3. (Hurwitz [Hu 1]). *The only normed division algebras are R, C, H and Ca.*

One can check directly that R, C, H and Ca are normed, though the calculation for Ca is somewhat lengthy. An alternative argument can be found in [*Cu*]. That there are no other normed division algebras follows from Propositions 5.1 and 5.2.

We end this section with the following description of all possible automorphisms of the Cayley numbers.

PROPOSITION 5.4. *Suppose* e_1, e_2 *and* e_3 *are orthonormal imaginary Cayley numbers with* e_3 *orthogonal to* $e_1 e_2$. *Then there exists a unique automorphism of* Ca *sending* $i = (i, 0) \mapsto e_1$, $j = (j, 0) \mapsto e_2$ *and* $\varepsilon = (0, 1) \mapsto e_3$.

This follows from three applications of Proposition 5.1.

From Proposition 5.4, one concludes that the group of all automorphisms of the Cayley numbers (a Lie group known as G_2) is 14-dimensional.

6. THE HOPF FIBRATION $S^7 \hookrightarrow S^{15} \to S^8$

Choose orthonormal coordinates in R^{16} and identify it with Cayley 2-space Ca^2. In Ca^2 consider subsets of the form

$$L_m = \{(u, mu) : u \in Ca\} \quad \text{for each} \quad m \in Ca,$$
$$L_\infty = \{(0, v) : v \in Ca\}.$$

They are 8-dimensional real linear subspaces of R^{16}, but *not* Cayley subspaces of Ca^2 because they are not closed under Cayley multiplication. This is the effect of the nonassociativity of the Cayley numbers. Nevertheless, we call L_m and L_∞ *Cayley lines* for simplicity.

We need to check that these Cayley lines fill out Ca^2, with any two meeting only at the origin. Given $(u, v) \in Ca^2$, if $u = 0$ then this point is on the Cayley line L_∞. If $u \neq 0$, let $m = v u^{-1}$. Then $m u = (v u^{-1}) u = v$ by Fact 3 of the preceding section. Hence the point (u, v) lies on the Cayley line L_m. Thus the Cayley lines fill out Ca^2.

Clearly L_∞ meets each other Cayley line only at the origin. And if the point (u, v), with $u \neq 0$, lies on the Cayley lines L_m and L_n, then $v = m u = n u$. Hence $m = n$. Thus any two Cayley lines meet only at the origin.

The unit 7-spheres on these Cayley lines then define for us the *Hopf fibration* $S^7 \hookrightarrow S^{15} \to S^8$. Note that the base space is clearly homeomorphic to an 8-sphere, since there is one Cayley line for each Cayley number m, and one for the number ∞.

In a similar fashion, if we start with any k-dimensional normed division algebra K, we obtain a Hopf fibration

$$S^{k-1} \hookrightarrow S^{2k-1} \to S^k.$$

Note by Hurwitz's theorem that K is isomorphic to R, C, H or Ca, so there are really no new cases.