§8. L. Kauffman's approach to V. Jones' one-variable polynomial

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of K. Hence, for a knot, a link with a single component, the exponent of m in $P_K(l, m)$ is even and therefore $\Delta_K(t) = P_K(i, i(t^{1/2} - t^{-1/2}))$ is indeed a Laurent polynomial in t.

To obtain the one-variable Jones polynomial we use the substitution $l = it$, $m = i(t^{1/2} - t^{-1/2})$. Explicitly,

$$V_K(t) = P_K(it, i(t^{1/2} - t^{-1/2}))$$

Then we have

PROPERTY 7.6. *$V_K(t)$ satisfies the skein invariance*

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0\,,$$

which (together with $V(\bigcirc)=1$) characterizes Jones *one-variable polynomial, with the sign conventions used in reference* [Jo_3].

Whereas $P_K(l, m)$ determines $\Delta_K(t)$ and $V_K(t)$, it is known that there are no other relations between these polynomials. More precisely:

(1) The Alexander polynomial $\Delta_K(t)$ does not determine Jones polynomial $V_K(t)$ because the trivial knot $\bigcirc$ and Conway's eleven crossing knot 11_{471} have $\Delta(t) = 1$, but $V_K(t) \neq 1$ for $K = 11_{471}$.

(2) $V_K(t)$ does not determine $\Delta_K(t)$: The knots 4_1 and 11_{388} have the same $V(t)$ but different $\Delta(t)$.

(3) $V_K(t)$ and $\Delta_K(t)$ together do not determine $P_K(l, m)$: The knot 11_{388} and its mirror image have the same $V(t)$ and $\Delta(t)$ but different $P(l, m)$.

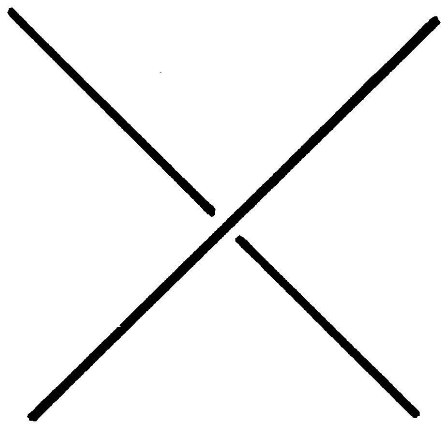
For more details on these questions, see [L.-M.].

We now turn to L. Kauffman's definition of the one-variable Jones polynomial $V_K(t)$ directly from the link diagram.

§ 8. L. Kauffman's approach to V. Jones' one-variable polynomial

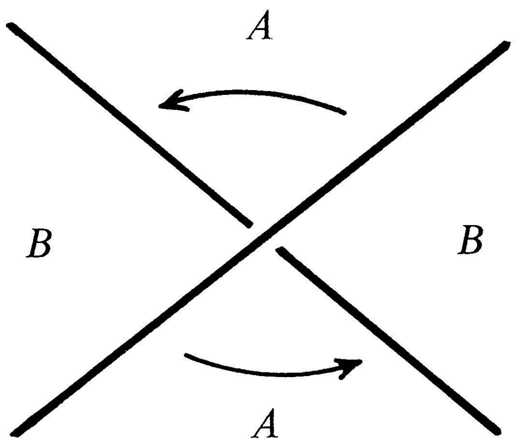
The importance of Kauffman's approach [Ka_3] is that it gives a new way to define and compute Jones polynomial $V_K(t)$. It is by using this definition that Kauffman and Murasugi prove their theorems about alternating links (see § 10 and 11).

Let L be an *unoriented* link diagram. Look at a double point; with no string orientation, they all look the same, up to a local homeomorphism:

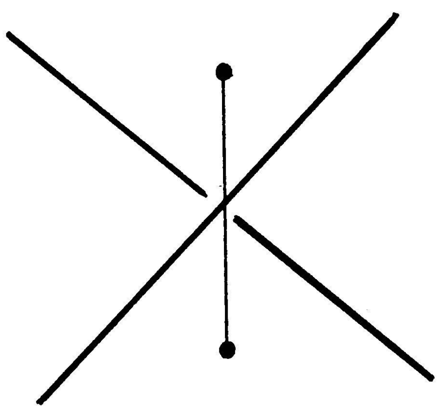


Locally, the plane $\mathbf{R}^2$ is divided into four regions.

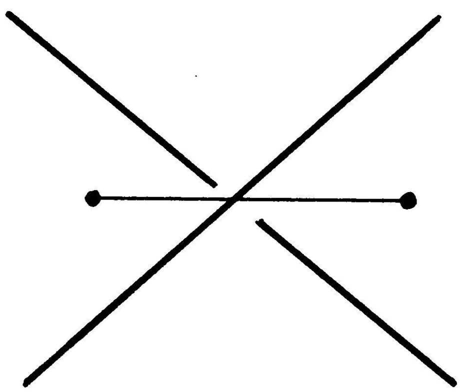
Look at the quarter turn the "over" line must make, in the positive sense, in order to coincide with the "under" line. Call "A" the two regions which are swept by the over line during the trip. Call "B" the other two.



Definition. A *marker* for a double point is a choice of "A" or "B" for this double point. It is symbolised like that:

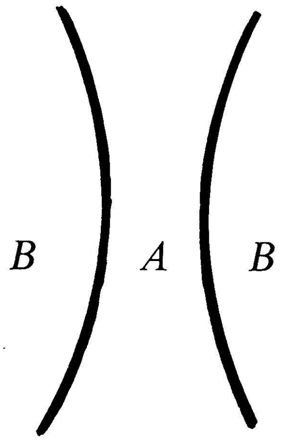


Marker A

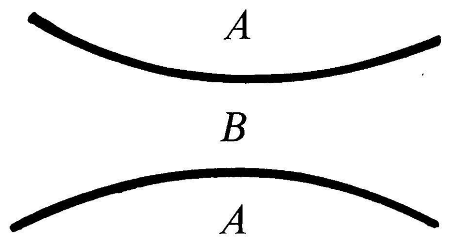


Marker B

Now, if a marker is chosen, one can split the link diagram by connecting the two opposite regions whose name has been elected. Here are the pictures.



splitting if marker
A is chosen



splitting if marker
B is chosen

Definition. A *state* S for L is a choice of a marker at every double point of L.

Suppose now that a state S for L is given. Make the correct splitting at every double point of L. The underlying knot projection Γ is transformed into a bunch Γ_S of disjoint simple closed curves in S^2. Let $|S|$ be the number of curves in Γ_S.

Write $a(S)$ for the number of markers A in the state S and write $b(S)$ for the number of B's.

If $c(L)$ denotes the number of crossings (double points) of L, one clearly has $2^{c(L)}$ states.

L being given, Kauffman defines a polynomial $<L> \in \mathbf{Z}[A, B, d]$ in the following way:

$$<L> = \sum_S A^{a(S)} B^{b(S)} d^{|S|-1}$$

the summation being taken over the $2^{c(L)}$ states.

Notations. Write "$\bigcirc$" for an unoriented, connected, simple closed curve in $\mathbf{R}^2$ and write $\bigcirc \amalg L$ for a disjoint union of such a diagram and an unoriented link diagram L.

Property 1. $<\bigcirc> = 1$.

Property 2. $<\bigcirc \amalg L> = d<L>$ if L is non empty.

Property 3. Let L be an unoriented link diagram. Select a crossing $\times$ and write L_A for the diagram obtained from L by connecting the two regions A at $\times$, and write L_B for the diagram obtained by connecting the two B's. Then:

$$<L> = A<L_A> + B<L_B> .$$

PROPOSITION 8.1. *$<\ >$ is the unique function from the set of unoriented link diagrams to $\mathbf{Z}[A, B, d]$ which satisfies properties 1, 2 and 3.*

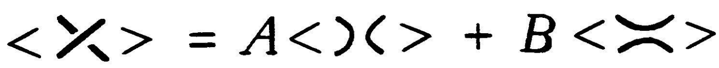
The proof is straightforward.

PROPOSITION 8.2. *If one sets $B = A^{-1}$ and $d = -(A^2+A^{-2})$, one gets a function into $\mathbf{Z}[A^{\pm 1}]$ which is invariant under Reidemeister moves (ii) and (iii).*

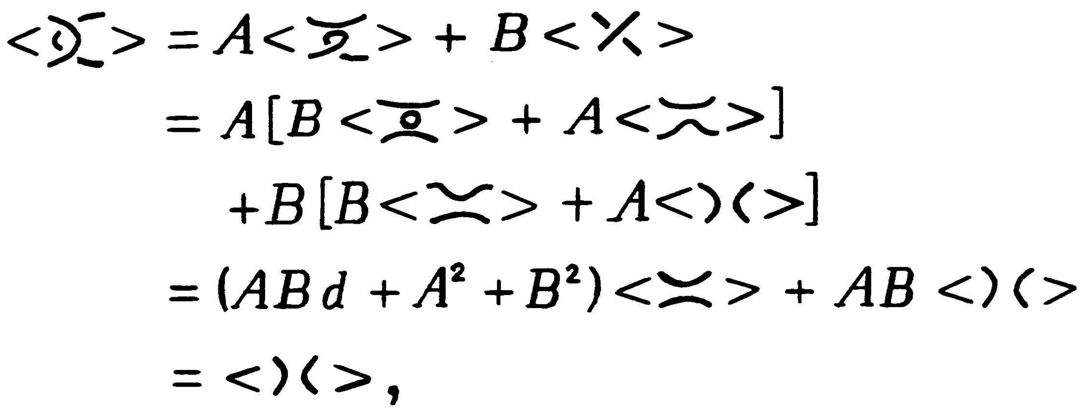
Notations. Following Kauffman, we shall still write $<\ >$ for the function into $\mathbf{Z}[A^{\pm 1}]$. From now on, only this function will be used.

We now recall briefly Kauffman's proof of proposition 8.2.

First of all, we shall use Kauffman's schematic way of writing property 3:

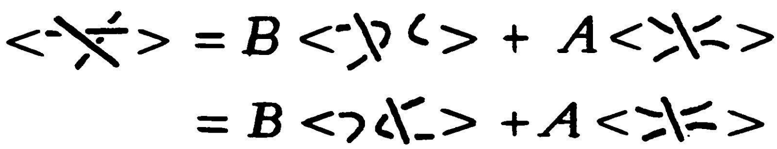


Invariance under move (ii):



since we have set $B = A^{-1}$ and $d = -(A^2+A^{-2})$.

Invariance under move (iii):



by invariance under move (ii)

$$= <$$  $$> .$$

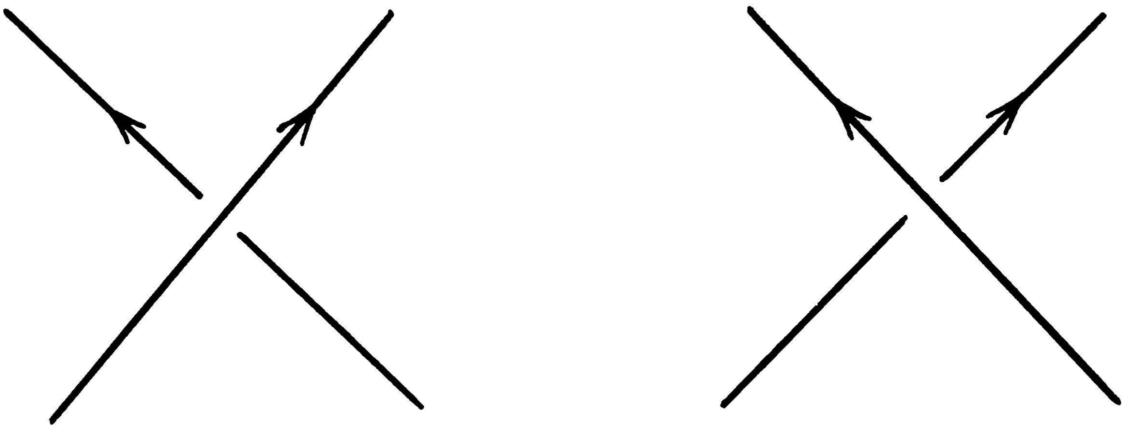
Q.E.D.

This seems to be as far as one can get without orienting link diagrams, because $<\ \ >$ is *not* invariant under Reidemeister move (i).

To remedy this state of affairs, Kauffman proceeds like this:

Let L be an *oriented* link diagram.

Recall now that, up to a rotation in $\mathbf{R}^2$, there are two types of double points:



Sign $+1$ Sign -1

Definition. The writhe number $w(L)$ is the sum of the signs of the double points of L.

This number is also called twist number. It was known to Tait and much used by Little. See § 9 of these notes.

Kauffman's polynomial $f_L(A) \in \mathbf{Z}[A^{\pm 1}]$ is then defined in the following way:

$$f_L(A) = (-A)^{-3w(L)} <L> .$$

PROPOSITION 8.3. *The polynomial* f *is invariant under Reidemeister moves (i), (ii) and (iii).*

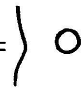
Proof of proposition 8.3. The writhe number is unchanged by the moves (ii) and (iii). Hence proposition (8.2) implies the invariance of f under the moves (ii) and (iii).

We now prove the invariance under move (i).

Let $\hat{L}$ be a link diagram with a portion looking like this:

and let L be the link diagram obtained from $\hat{L}$ by removing the loop.

It is immediate that

$\hat{L}_A =$  and $\hat{L}_B =$ 

If we apply property 3 for $< \; >$ we get

$$<\hat{L}> \;=\; A<L> \;+\; A^{-1}<L \amalg \bigcirc> .$$

By property 2

$$<\hat{L}> \;=\; A<L> \;+\; A^{-1}(-A^2-A^{-2}) <L> .$$

So $<\hat{L}> \;=\; \big(A-A^{-1}(A^2+A^{-2})\big) <L> \;=\; (-A)^{-3} <L>$.

Now, for any orientation of the string, the sign of the double point is -1.

Hence $w(\hat{L}) = w(L) - 1$.

Going back to the definition,

$$\begin{aligned} f_{\hat{L}} &= (-A)^{-3w(\hat{L})} <\hat{L}> \;=\; (-A)^{-3w(L)+3} <\hat{L}> \\ &= (-A)^{-3w(L)+3} (-A)^{-3} <L> \\ &= (-A)^{-3w(L)} <L> \;=\; f_L . \end{aligned}$$

The proof for the other loop is similar. Q.E.D.

From proposition 8.3 we deduce that Kauffman's polynomial induces a map $f: \mathscr{L} \to \mathbf{Z}[A^{\pm 1}]$.

THEOREM 8.4. *The map* $f: \mathscr{L} \to \mathbf{Z}[A^{\pm 1}]$ *satisfies:*

1. $f(\bigcirc) \equiv 1$.
2. *If* L_+, L_- *and* L_0 *are skein related (see § 3), then:*

$$A^4 f_{L_+} - A^{-4} f_{L_-} = (A^{-2}-A^2) f_{L_0} .$$

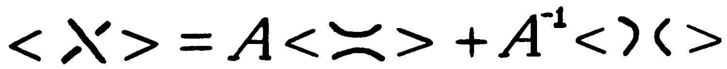
From the universality of Jones polynomial, we obtain:

COROLLARY 8.5. *Let* K *be an oriented link in* $\mathbf{R}^3$ *and let* L *be an oriented diagram of* K. *Then:*

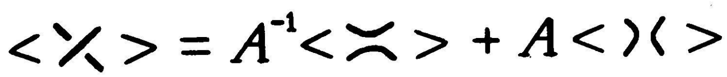
$$V_K(t) = f_L(t^{1/4}).$$

Recall that we use Jones definition in the Bulletin AMS [Jo_3] for V_K. If we were to use Jones definition in the Notices AMS [Jo_4], we would set $A = t^{-1/4}$.

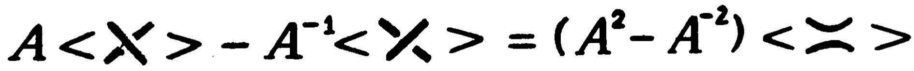
Proof of theorem 8.4. The proof of 1. is straightforward from the definition. For 2., using Kauffman's notations one has:



and



Hence:



If we orient the strings and put the writhe number in the picture, we get the formula 2. Q.E.D.

Using L. Kauffman's definition of Jones polynomial, the following properties are easily proved (enjoyable exercise left to the reader):

I. If K_1 and K_2 are two oriented links in S^3, let $K_1 \amalg K_2$ denote their distant union (one in each hemisphere). Then:

$$V_{K_1 \amalg K_2} = \mu\, V_{K_1} \cdot V_{K_2}$$

where $\mu = -(t^{1/2} + t^{-1/2})$.

II. Let $K_1 \# K_2$ denote any connected sum of K_1 and K_2 as in § 7 prop. 4. Then:

$$V_{K_1 \# K_2} = V_{K_1} \cdot V_{K_2}.$$

III. Let $K^{\times}$ denote the mirror image of K. Then:

$$V_{K^{\times}}(t) = V_K(t^{-1}).$$

The first three formulas are rather straightforward from the definitions.

IV. (Jones reversing result). Let K be an oriented link in S^3 and let γ be a component of K. Let λ be the linking coefficient of γ with what is left of K when we remove γ. (We suppose that this is not empty!) Let $\hat{K}$ be the oriented link obtained from K by changing the orientation of γ, while keeping the others fixed. Then:

$$V_{\hat{K}}(t) = t^{3\lambda} V_K(t).$$

Proof. Of course, we have $\langle K \rangle = \langle \hat{K} \rangle$, because, for the polynomial $\langle\ \rangle$, orientations do not matter.
Now: $w(K) = w(\gamma) + 2\lambda$.
So: $w(\hat{K}) = w(\gamma) - 2\lambda$.
Hence: $w(\hat{K}) = w(K) - 4\lambda$.
We substitute and get:

$$f_{\hat{K}}(A) = (-A)^{-3w(\hat{K})} \langle \hat{K} \rangle = (-A)^{-3w(K)+12\lambda} \langle K \rangle$$
$$= (-A)^{12\lambda}(-A)^{-3w(K)} \langle K \rangle = (-A)^{12\lambda} f_K(A) = A^{12\lambda} f_K(A).$$

As one substitutes $t^{1/4}$ for A to get Jones 1-variable polynomial, the result follows.

To finish this paragraph, we illustrate quickly Kauffman's definition by computing Jones one variable polynomial for the right-handed trefoil T_+. (Compare § 3.)

There are 8 states associated to the standard knot diagram. One readily sees that

$$\langle T_+ \rangle = A^3 d + 3A^2 B d^0 + 3AB^2 d + B^3 d^2.$$

Substituting $d = -(A^2 + A^{-2})$ and $B = A^{-1}$ one gets

$$\langle T_+ \rangle = -A^5 - A^{-3} + A^{-7}.$$

As $w(T_+) = 3$, one gets

$$f_{T_+}(A) = (-A)^{-9} \langle T_+ \rangle = A^{-4} + A^{-12} - A^{-16}.$$

Substituting $t = A^{1/4}$ one finally obtains

$$V_{T_+}(t) = t^{-1} + t^{-3} - t^{-4} = t^{-4}(-1 + t + t^3).$$

Now, if one uses our computation in § 3

$$P(T_+) = -2a_-a_+^{-1} - a_-^2a_+^{-2} + a_+^{-2}a_0^2$$

and substitutes $a_+ = l, a_- = l^{-1}, a_0 = m$ one gets

$$P_{T_+}(l, m) = (-2l^{-2}-l^{-4})m^0 + l^{-2}m^2 .$$

The last substitution $l = it$; $m = i(t^{1/2}-t^{-1/2})$ gives (with relief!) the same result for Jones one variable polynomial. (Bulletin AMS definition.)

§ 9. Tait conjectures

Tait was primarily interested in the classification of knots (i.e. one component links). He organized the job in two steps.

Step 1. Classify generic immersions of the circle in S^2 (not $\mathbf{R}^2$!) modulo homeomorphisms (possibly orientation reversing) of S^2. This was mostly done by the Rev. T. P. Kirkman (around 1880).

In this process, one has to remember that one is looking at knots in $\mathbf{R}^3$ and that one is trying to list knots according to their "knottiness", i.e. their minimal crossing number. So, Tait first reduced the number of double points of a generic immersion by making one "local 180° rotation".

Examples.