# §5. The trace

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The fundamental idea of V. Jones which led him to the definition of his original one-variable polynomial is the construction of the trace. Originally, V. Jones used algebras which are quotients of the algebras $H_n$. The lifting of the trace to the Hecke algebras $H_n$ was observed by A. Ocneanu.

The trace will commute with the inclusion $H_n \to H_{n+1}$ and therefore yield a trace on the direct limit of the $H_n$'s. (Compare with the discussion in § 12.)

Theorem. *Let $K$ be a field and let $q, z \in K$ be two elements of $K$. Let $H_n$ be the* Hecke *algebra over $K$ corresponding to $q$. There exists a trace $\mathrm{Tr}: H_n \to K$ compatible with the inclusion $H_n \to H_{n+1}$,* i.e. *the diagram*

$$\begin{tikzcd} H_n \arrow[rr] \arrow[dr, "\mathrm{Tr}"'] & & H_{n+1} \arrow[dl, "\mathrm{Tr}"] \\ & K & \end{tikzcd}$$

*commutes, and such that*

(1) $\mathrm{Tr}(1) = 1$,

(2) $\mathrm{Tr}$ *is $K$-linear and* $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$,

(3) *If* $a, b \in H_n$, *then* $\mathrm{Tr}(aT_nb) = z\mathrm{Tr}(ab)$.

Notice that the last property enables us to calculate $\mathrm{Tr}(x)$ for an arbitrary $x \in H_n$ by using the fact that monomials in normal form generate $H_n$ over $K$. For instance,

$$\mathrm{Tr}(T_1) = z,$$
$$\mathrm{Tr}(T_1T_2) = \mathrm{Tr}(T_2T_1) = z^2,$$
$$\mathrm{Tr}(T_1T_2T_1) = z\mathrm{Tr}(T_1^2) = z\big((q-1)z+q\big).$$

*Proof.* The $K$-linear map $\mathrm{Tr}: H_{n+1} \to K$ is defined by induction on $n$, using the structure lemma of § 4 (Proposition 4.1):

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \xrightarrow{\sim} H_{n+1}.$$

Starting with $\mathrm{Tr}\colon H_0 = K \to K$ the identity, one defines $\mathrm{Tr}\colon H_{n+1} \to K$ by $\mathrm{Tr}(x) = \mathrm{Tr}(a) + \Sigma_i\, z\mathrm{Tr}(b_i c_i)$, if $\varphi(a+\Sigma_i b_i \otimes c_i) = x$.

It is clear that if $a, b \in H_n$, then

$$\mathrm{Tr}(aT_n b) = z\mathrm{Tr}(ab)\,,$$

since $\varphi(a\otimes b) = aT_n b$.

The only statement to be proved is then:

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) \quad \text{for all} \quad x, y \in H_{n+1}\,.$$

This is proved by induction on $n$.

We may assume that $x$ and $y$ are monomials containing $T_n$ at most once.

If $y$ does not contain $T_n$ at all, then writing $x = x'T_n x''$, where $x', x''$ are monomials in $T_1, \ldots, T_{n-1}$, one has

$$\mathrm{Tr}(xy) = z\mathrm{Tr}(x'x''y) = z\mathrm{Tr}(yx'x'') = \mathrm{Tr}(yx'T_n x'') = \mathrm{Tr}(yx)\,.$$

If $y$ contains $T_n$, it suffices to check the case where $x = aT_n b$ and $y = T_n$, as is easily verified. (Here $a, b \in H_n$.)

There are various cases depending on whether or not the elements $a$ and $b$ actually contain $T_{n-1}$. The worst case is the one in which $a = a'T_{n-1}a''$, $b = b'T_{n-1}b''$ with $a', a'', b', b''$ belonging to $H_{n-1}$. We have then

$$\begin{aligned}\mathrm{Tr}(aT_n bT_n) &= z\big((q-1)\mathrm{Tr}(ab) + q\mathrm{Tr}(ab'b'')\big)\\ \mathrm{Tr}(T_n aT_n b) &= z\big((q-1)\mathrm{Tr}(ab) + q\mathrm{Tr}(a'a''b)\big)\,.\end{aligned}$$

But

$$\mathrm{Tr}(ab'b'') = \mathrm{Tr}(a'T_{n-1}a''b'b'') = z\mathrm{Tr}(a'a''b'b'')\,,$$

and

$$\mathrm{Tr}(a'a''b) = \mathrm{Tr}(a'a''b'T_{n-1}b'') = z\mathrm{Tr}(a'a''b'b'')\,.$$

Hence,

$$\mathrm{Tr}(aT_n bT_n) = \mathrm{Tr}(T_n aT_n b)$$

as desired.