

# §7. Quantized Contact Transformations

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An analytic subset  $V$  of  $T^*X$  is called *involutive* if  $f|_V = g|_V = 0$  implies  $\{f, g\}|_V = 0$ .

The following theorem exhibits a phenomenon which has no analogue in the commutative case.

**THEOREM 6.3.2 ([G]).** *Let  $\mathcal{M}$  be a coherent  $\mathcal{E}_X$ -module defined on an open subset  $\Omega$  of  $T^*X$  and let  $\mathcal{L}$  be a  $\mathcal{E}_X(0)|_\Omega$ -module which is a union of coherent  $\mathcal{E}_X(0)$ -modules. Then  $V = \{p \in \Omega; \mathcal{L} \text{ is not coherent over } \mathcal{E}_X(0) \text{ on any neighborhood of } p\}$  is an involutive analytic subset of  $\Omega$ .*

**COROLLARY 6.3.3 ([SKK] Chap. II, Theorem 5.3.2, [M]).** *For any coherent  $\mathcal{E}_X$ -module  $\mathcal{M}$ ,  $\text{Supp } \mathcal{M}$  is involutive.*

Since any involutive subset has codimension less than or equal to  $\dim X$ , we have

**COROLLARY 6.3.4.** *The support of a coherent  $\mathcal{E}_X$ -module has codimension  $\leq \dim X$ .*

After some algebraic calculation, this implies

**THEOREM 6.3.5 ([SKK] Chap. II, Theorem 5.3.5).** *For any point  $p \in T^*X$ ,  $\mathcal{E}_{X,p}$  has a global cohomological dimension  $\dim X$ .*

6.4. An analytic subset  $\Lambda$  of  $T^*X$  is called *Lagrangian* if  $\Lambda$  is involutive and  $\dim \Lambda = \dim X$ . A coherent  $\mathcal{E}_X$ -module is called *holonomic* if its support is Lagrangian.

## § 7. QUANTIZED CONTACT TRANSFORMATIONS

7.1. In the previous section, we saw that the symplectic structure of  $T^*X$  is closely related to micro-differential operators via the relation of commutator and Poisson bracket. In this section, we shall explain another relation.

*Definition 7.2.1.* Let  $X$  and  $Y$  be complex manifolds of the same dimension. A morphism  $\varphi$  from an open subset  $U$  of  $T^*X$  to  $T^*Y$  is called a *homogeneous symplectic transformation* if  $\varphi^*\theta_Y = \theta_X$ .

We can easily see the following

(7.2.1) If  $\varphi$  is a homogeneous symplectic transformation, then  $\varphi$  is a local isomorphism and is compatible with the action of  $\mathbf{C}^*$ .

(7.2.2) Assume  $Y = \mathbf{C}^n$  and let  $(y_1, \dots, y_n; \eta_1, \dots, \eta_n)$  be the coordinates of  $T^*Y$ , so that  $\theta_Y = \sum \eta_j dy_j$ .

Set  $p_j = \eta_j \circ \varphi$  and  $q_j = y_j \circ \varphi$ . Then we have

(7.2.3.1)  $\{p_j, p_k\} = \{q_j, q_k\} = 0, \{p_j, q_k\} = \delta_{j,k}$  for  $j, k = 1, \dots, n$ .

(7.2.3.2)  $p_j$  is homogeneous of degree 1 and  $q_j$  is homogeneous of degree 0 with respect to the fiber coordinates.

(7.2.4) Conversely assume that functions  $\{q_1, \dots, q_n, p_1, \dots, p_n\}$  on  $U \subset T^*X$  satisfy (7.2.3.1) and (7.2.3.2). Then the map  $\varphi: U \rightarrow T^*Y$ , given by

$$U \ni x \mapsto (q_1(x), \dots, q_n(x); p_1(x), \dots, p_n(x)) \in T^*Y,$$

is a homogeneous symplectic transformation. We call  $(q_1, \dots, q_n; p_1, \dots, p_n)$  a *homogeneous symplectic coordinate system*.

**THEOREM 7.2.2** ([SKK] Chap. II § 3.2, [K2] § 2.4, [Bj] Chap. 4 § 6).

Let  $\varphi: T^*X \supset U \rightarrow T^*Y$  be a homogeneous symplectic transformation, let  $p_X$  be a point of  $U$  and set  $p_Y = \varphi(p_X)$ . Then we have

- (a) There exists an open neighborhood  $U'$  of  $p_X$  and a  $\mathbf{C}$ -algebra isomorphism  $\Phi: \varphi^{-1}\mathcal{E}_Y|_{U'} \xrightarrow{\sim} \mathcal{E}_X|_{U'}$  (we call  $(\varphi, \Phi)$  a *quantized contact transformation*).
- (b) If  $\Phi: \varphi^{-1}\mathcal{E}_Y \rightarrow \mathcal{E}_X|_U$  is a  $\mathbf{C}$ -algebra homomorphism then for any  $m, \Phi$  gives an isomorphism  $\varphi^{-1}\mathcal{E}_Y(m) \xrightarrow{\sim} \mathcal{E}_X(m)|_U$ . Moreover the following diagram commutes:

$$\begin{array}{ccc} \varphi^{-1}\mathcal{E}_Y(m) & \xrightarrow{\Phi} & \mathcal{E}_X(m)|_U \\ \downarrow \sigma_m & & \downarrow \sigma_m \\ \varphi^{-1}\mathcal{O}_{T^*Y}(m) & \xrightarrow{\varphi^*} & \mathcal{O}_{T^*Y}(m)|_U \end{array}$$

- (c) Let  $\Phi$  and  $\Phi'$  be two  $\mathbf{C}$ -algebra homomorphisms  $\varphi^{-1}\mathcal{E}_Y \rightarrow \mathcal{E}_X|_U$ .

Then there exist  $\lambda \in \mathbf{C}$ , a neighborhood  $U'$  of  $p_X$  and  $P \in \Gamma(U; \mathcal{E}_X(\lambda))$  such that  $\sigma_\lambda(P)$  is invertible and

$$\Phi'(Q) = P\Phi(Q)P^{-1} \quad \text{for } Q \in \varphi^{-1}\mathcal{E}_Y|_{U'}.$$

Moreover  $\lambda$  is unique and  $P$  is unique up to constant multiple.

(d) Let  $Y = \mathbf{C}^n$  and let  $U$  be an open subset of  $T^*X$ .

If  $P_j \in \Gamma(U; \mathcal{E}_X(1))$  and  $Q_j \in \Gamma(U; \mathcal{E}_X(0))$  ( $1 \leq j \leq n$ ) satisfy

$$(7.2.5) \quad \begin{aligned} [P_j, P_k] &= [Q_j, Q_k] = 0 \\ [P_j, Q_k] &= \delta_{jk} \end{aligned}$$

then there exists a unique quantized contact transformation  $(\varphi, \Phi)$  such that

$$\varphi(p) = (\sigma_0(Q_1)(p), \dots, \sigma_0(Q_n)(p), \sigma_1(P_1)(p), \dots, \sigma_1(P_n)(p)),$$

and  $\Phi(y_j) = Q_j, \Phi(\partial_{y_j}) = P_j$ .

We call  $\{Q_1, \dots, Q_n, P_1, \dots, P_n\}$  quantized canonical coordinates.

7.3. We shall give several examples of quantized contact transformations.

*Example 7.3.1.* If  $P(\partial)$  is a constant coefficient micro-differential operator of order 1, then

$$(x_1 + [P, x_1], x_2 + [P, x_2], \dots, x_n + [P, x_n], \partial_{x_1}, \dots, \partial_{x_n})$$

gives quantized canonical coordinates.

*Example 7.3.2.* More generally if  $P$  is a micro-differential operator of order 1 and  $\exp tH_{\sigma_1(P)}$  exists, then  $\exp tP$  gives a quantized contact transformation  $\Phi_t$ , by solving the equation  $\frac{d}{dt}\Phi_t(Q) = [P, \Phi_t(Q)]$  with the initial condition  $\Phi_t(Q) = Q$  for  $t = 0$ .

*Example 7.3.3.* (Paraboloidal transformation [K2] p. 36). Set  $X = \mathbf{C}^{1+n} = \{(t, x) \in \mathbf{C} \times \mathbf{C}^n\}$ ,

$$\Omega = \{(t, x; \tau, \xi) \in T^*X; \tau \neq 0\}, G = \text{Sp}(n; \mathbf{C})$$

$$= \{g \in \text{GL}(2n; \mathbf{C}); {}^t g J g = J\} \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ , let  $\Psi_g$  be the quantized contact transformation given by

$$\partial_x \mapsto \alpha \partial_x - \beta x \partial_t$$

$$x \mapsto \gamma \partial_x \partial_t^{-1} + \delta x$$

$$\partial_t \mapsto \partial_t$$

$$t \mapsto t + \frac{1}{2} \{ \langle \partial_x, {}^t \gamma \alpha \partial_x \rangle \partial_t^{-2} + \langle \partial_x, {}^t \gamma \beta x \rangle \partial_t^{-1} \\ + \langle {}^t \gamma \beta x, \partial_x \rangle \partial_t^{-1} + \langle x, {}^t \delta \beta x \rangle \}.$$

Then we have  $\Psi_{g_1} \Psi_{g_2} = \Psi_{g_1 g_2}$ .

## § 8. FUNCTORIAL PROPERTIES OF MICRO-DIFFERENTIAL MODULES (See [SKK])

### 8.1. External Tensor Product.

Let  $X$  and  $Y$  be complex manifolds and let  $p_1$  and  $p_2$  be the projections  $T^*(X \times Y) \rightarrow T^*X$  and  $T^*(X \times Y) \rightarrow T^*Y$ , respectively. Then  $\mathcal{E}_{X \times Y}$  contains  $p_1^{-1} \mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1} \mathcal{E}_Y$  as a subring. For an  $\mathcal{E}_X$ -module  $\mathcal{M}$  and an  $\mathcal{E}_Y$ -module  $\mathcal{N}$ , we define the  $\mathcal{E}_{X \times Y}$ -module  $\mathcal{M} \hat{\otimes} \mathcal{N}$  by

$$(8.1.1) \quad \mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{E}_{X \times Y} \otimes_{p_1^{-1} \mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1} \mathcal{E}_Y} (p_1^{-1} \mathcal{M} \otimes_{\mathbb{C}} p_2^{-1} \mathcal{N}).$$

Then one can easily see

#### PROPOSITION 8.1.1.

- (i)  $\mathcal{M} \hat{\otimes} \mathcal{N}$  is an exact functor in  $\mathcal{M}$  and in  $\mathcal{N}$  and  $\text{Supp}(\mathcal{M} \hat{\otimes} \mathcal{N}) = \text{Supp} \mathcal{M} \times \text{Supp} \mathcal{N}$ .
- (ii) If  $\mathcal{M}$  is  $\mathcal{E}_X$ -coherent and  $\mathcal{N}$  is  $\mathcal{E}_Y$ -coherent, then  $\mathcal{M} \hat{\otimes} \mathcal{N}$  is  $\mathcal{E}_{X \times Y}$ -coherent.

8.2. For a complex submanifold  $Y$  of a complex manifold  $X$  of codimension  $l$ , the sheaf  $\lim_{\rightarrow m} \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_X/\mathcal{I}^m, \mathcal{O}_X)$  has a natural structure of  $\mathcal{D}_X$ -module,

which is denoted by  $\mathcal{B}_{Y|X}$ . Here  $\mathcal{I}$  is the defining ideal of  $Y$ . The homomorphism  $\mathcal{O}_Y \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_Y, \Omega_X^l) \rightarrow \Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$  gives the canonical section  $c(Y, X)$  of  $\Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$ . If we take local coordinates  $(x_1, \dots, x_n)$  of  $X$  such that  $Y$  is defined by  $x_1 = \dots = x_l = 0$ , then we have