1. The Milnor Lattice of a Singularity

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The results were obtained by the following method:

a) Find a distinguished basis for the given singularity. This is done using methods of Gabrielov, especially [11]. But the Dynkin diagrams of these bases are very complicated and contain many cycles.

b) Transform this diagram into a "nicer" form, where the information one is looking for is more transparent.

c) Analyse this diagram.

The paper is organized as follows. In section 1 + 2 we recall the definitions of the invariants and the basic relations among them and discuss the admissible transformations for b) above. Section 3 is devoted to a study of the weaker invariants including weakly distinguished bases of the above mentioned singularities. In section 4 we consider distinguished bases of the uni- and bimodular singularities.

There are also other invariants associated to singularities such as e.g. the monodromy groups. For a discussion of the relations among the various invariants and of their relative strength with respect to geometrical problems we refer to the expository article of E. Brieskorn [3]. For a description of the monodromy groups we refer to [8, 9]. There are also other interesting phenomena related to the above invariants in the class of bimodular singularities such as an extension of Arnold's strange duality. This is the subject of a joint paper with C. T. C. Wall, which is in preparation.

This paper is an extended version of the talk given by the author at the conference on the "Topology of complex singularities" at Les Plans-sur-Bex/Switzerland, March 27-April 2, 1982. The author thanks the organizers of this meeting, especially C. Weber, for their invitation. He is also grateful to E. Brieskorn for helpful discussions, which influenced the presentation in section 2. The final work on this paper was carried out at the State University of Utrecht and was supported by the Netherlands Foundation for Mathematics S.M.C. with financial aid from the Netherlands Organization for the Advancement of Pure Research (Z.W.O.). The author thanks these institutions for their hospitality.

1. The Milnor Lattice of a Singularity

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of an analytic function with an isolated singularity at 0. Let B_ε denote an open ball of radius ε in $\mathbf{C}^n$ around 0. Then for sufficiently small $\delta > 0$ and $\varepsilon > 0$

$$V_\delta = f^{-1}(\delta) \cap B_\varepsilon$$

is the Milnor fiber of f. The Milnor fiber has the homotopy type of a bouquet of $n-1$-spheres, and therefore its only interesting homology group is

$$L = H_{n-1}(V_\delta, \mathbf{Z}),$$

a free $\mathbf{Z}$-module of rank μ. We shall assume throughout this paper that $n \equiv 3(4)$. Then the intersection form $\langle\ ,\ \rangle$ on L is symmetric and satisfies $\langle x, x\rangle \in 2\mathbf{Z}$ for all $x \in L$. Therefore L provided with $\langle\ ,\ \rangle$ is an even lattice, which we call the *Milnor lattice* of f. To L is associated a triple of numbers (μ_0, μ_+, μ_-), where μ_0, μ_+, μ_- is the number of 0's, 1's, -1's respectively on the diagonal after a diagonalisation of the quadratic form over the real numbers.

To L is also associated another invariant, which we define next. Let ker L denote the kernel of L and $\bar{L} = L/\ker L$ the corresponding nondegenerate lattice. Let $\bar{L}^{\#} = \mathrm{Hom}(\bar{L}, \mathbf{Z})$ be the dual lattice of $\bar{L}$, and $G_L = \bar{L}^{\#}/\bar{L}$. Then G_L is a finite abelian group of order $|\operatorname{disc} \bar{L}|$. The bilinear form on L induces a bilinear form

$$b_L \colon G_L \times G_L \to \mathbf{Q}/\mathbf{Z}$$

on G_L defined by $b_L(\bar{u}, \bar{v}) = \langle u, v\rangle \pmod 1$ for $u, v \in \bar{L}^{\#}$. This form is called *discriminant bilinear form.* Since L is even, there is also an induced quadratic form

$$q_L \colon G_L \to \mathbf{Q}/2\mathbf{Z}$$

defined by $q_L(\bar{u}) = \langle u, u\rangle \pmod 2$ for $u \in \bar{L}^{\#}$. It is called *discriminant quadratic form.* The pair (G_L, b_L) can be interpreted geometrically as follows: Let K denote the link of the singularity f, which is equal to the boundary of $\bar{V}_\delta$,

$$K = \partial \bar{B}_\varepsilon \cap f^{-1}(0) = \partial \bar{V}_\delta .$$

Then G_L is the torsion subgroup of $H_{n-2}(K, \mathbf{Z})$ and b_L is the classical linking form (cf. [4]). Moreover μ_0 is equal to the rank of $H_{n-2}(K, \mathbf{Z})$.

There are results in the theory of quadratic forms of Durfee, Kneser, Nikulin and Wall, that the above invariants are already sufficient to determine the isomorphism class of the lattice, if the lattice satisfies certain conditions, in particular is indefinite. From these results V. V. Nikulin has derived the following theorem (cf. [17]).

THEOREM 1.1. *The Milnor lattice L is determined as an abstract lattice by (μ_0, μ_+, μ_-) and the discriminant bilinear form b_L (resp. the discriminant quadratic form q_L).*