§3. Blowing Down

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 $\mathscr{S}_{\text{Ab}}(V)$ is the set of distinct algebraic structures on V. Hence a natural problem is to compute $\mathscr{S}_{\mathrm{Alg}}(V)$, or at least produce nontrivial elements of this set. For example if we take $M \subset V$ as in Proposition 2.10, then by Theorem 2.12 (V, M) is diffeomorphic to nonsingular algebraic sets (V', M') . Let $\vert V \vert = \vert V' \vert$ denote the underlying smooth structures and let $V \stackrel{g}{\rightarrow} |V|$, $V' \stackrel{g'}{\rightarrow} |V|$ be the forgetful maps. Then (V, g) and (V', g') are distinct elements of $\mathcal{S}_{Alg}(|V|)$, otherwise M would be isotopic to ^a nonsingular algebraic subset of V

An interesting question is whether algebraic structures on smooth manifolds satisfy the product structure theorem ; that is, whether the natural map

$$
\mathscr{S}_{\mathrm{Alg}}(M) \times \mathbf{R}^n \to \mathscr{S}_{\mathrm{Alg}}(M \times \mathbf{R}^n), (V, g) \mapsto (V \times \mathbf{R}^n, g \times id)
$$

is surjection. The answer would be negative if one can find a smooth manifold M and $\theta \in H_{\mu}(M; \mathbb{Z}/2\mathbb{Z})$ such that M can not be diffeomorphic to a nonsingular algebraic set M' with $\theta \in H^A_*(M' ; \mathbb{Z}/2\mathbb{Z})$. To see this, pick any smooth representative $N \stackrel{g}{\rightarrow} M$ of $\theta = g_*[N]$. By graphing g, we can assume $N \subset M$ \times Rⁿ for some *n* and *g* is induced by projection. By Theorem 2.12 we can find a diffeomorphism $\lambda : M \times \mathbb{R}^n \to V$ to a nonsingular algebraic set V with $\lambda(N)$ is an algebraic subset (one has to modify Theorem 2.12 to apply to this noncompact case). Then there can not exist a birational diffeomorphism $\mu: V$ $M' \times \mathbb{R}^n$ where M' is a nonsingular algebraic set diffeomorphic to M, otherwise $\lambda(N) \stackrel{\mu}{\rightarrow} M' \times \mathbf{R}^n \stackrel{\text{projection}}{\longrightarrow} M'$ would represent $\theta \in H^A_*(M'; \mathbf{Z}/2\mathbf{Z})$.

§3. Blowing Down

Real algebraic sets obey some simple but useful topological properties :

PROPOSITION 3.1.

- (a) One point compactification an algebraic set is homeomorphic to an algebraic set.
- (b) Given algebraic sets $L \subset V$, then $V L$ is homeomorphic to an algebraic set.
- (c) Given algebraic sets $L \subset V$ with V compact then V/L is homeomorphic to an algebraic set.

Proof:

(a) Let $Z \subset \mathbb{R}^n$ be an algebraic set and assume that $Z \neq \mathbb{R}^n$ and $0 \notin Z$ (otherwise translate Z). Let $Z = f^{-1}(0)$ for some polynomial $f(x)$; then define $F(x) = |x|^{2d} f\left(\frac{x}{|x|^{2}}\right)$, where d is the degree of $f(x)$. Clearly $F(x)$ is a polynomial and $F^{-1}(0)$ is the one point compactification of Z, since $x \mapsto \frac{x}{1-x^2}$ is the $\vert x \vert$ inversion through the unit sphere.

(b) Let $V = f^{-1}(0)$, $L = g^{-1}(0)$ for some polynomials f, $g: \mathbf{R}^n \to \mathbf{R}$. Define $G(x, t) = |f(x)|^2 + |tg(x) - 1|^2$, then $G^{-1}(0) \approx V - L$.

(c) By applying (a) we get the one point compactification of $G^{-1}(0)$ to be an algebraic set; if V is compact this set is homeomorphic to V/L . \Box

This proposition implies that ^a set is homeomorphic to an algebraic set if and only if the one point compactification is homeomorphic to an algebraic set. Hence any noncompact algebraic set has ^a collar at infinity, since every algebraic set is locally cone-like [M]. Also we get that the reduced suspension $\Sigma^n X = X$ \times Sⁿ/X \vee Sⁿ of any algebraic set X is homeomorphic to an algebraic set.

There is a fancier version of the blowing down operation (c) (Proposition 3.3). First we need to discuss projectively closed algebraic sets. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Another interpretation of this concept is the following: Let λ : \mathbb{R}^n d. We call $p(x)$ an *overt polynomial* if $p_d^{-1}(0)$ is either the empty set or $\{0\}$. We call an algebraic set $V = p^{-1}(0)$ a projectively closed algebraic set if $p(x)$ is an overt polynomial. Another interpretation of this concept is the following: Let $\lambda : \mathbb{R}^n$ polynomial. Another interpretation of this concept is the following: Let $\lambda : \mathbf{R}^{-1}$
 $\rightarrow \mathbf{R}\mathbf{P}^{n}$ be the inclusion $\lambda(x_1, ..., x_n) = [1; x_1; ...; x_n]$ then $V = p^{-1}(0)$ is

projectively closed if and only if λ is a proje projectively closed if and only if λ is a projective algebraic subset of **RP**ⁿ in other words $\lambda(V)$ is Zariski closed in RPⁿ (see also [AK₂]). Real algebraic sets along with maps can easily be made projectively closed by the following.

PROPOSITION 3.2. Let $f: Z \to W$ be an entire rational function between algebraic sets with Z nonsingular and compact. Then there is a projectively closed algebraic set $V \subset W \times \mathbb{R}^n$ a birational diffeomorphism g which makes the following commute

$$
V \hookrightarrow W \times \mathbf{R}^n
$$
\n
$$
g \uparrow \approx \qquad \downarrow \pi
$$
\n
$$
Z \qquad \xrightarrow{f} \qquad W
$$

where π is the projection, n is some integer.

Proof: By taking the graph of f we can assume that $Z \subset W \times \mathbb{R}^m \subset \mathbb{R}^r$ for some r, and f is induced by projection. Also identify $\mathbf{R}^r \subset \mathbf{R}\mathbf{P}^r$ via λ . Then let \overline{Z} be the Zariski closure of Z in RP'. We claim $\dim(\overline{Z} - Z) < \dim(Z)$. This is because if U is an irreducible component of \overline{Z} then $U \cap Z \neq \emptyset$, and therefore $U - Z = U \cap \mathbb{RP}^{r-1}$ is a proper algebraic subset of U where \mathbb{RP}^{r-1} $=\{[0; x_1; ...; x_r] \in \mathbb{RP}^r\}$. Since U is irreducible dim $(U-Z) < dim(U)$, also $\dim(U) = \dim(Z)$. Therefore $\dim(\bar{Z} - Z) < \dim(Z)$. So $\bar{Z} - Z = \text{Sing}(\bar{Z})$. By resolution of singularities [H] (Theorem 1.1) there is ^a nonsingular algebraic set $V \subset \mathbb{RP}^r \times \prod \mathbb{RP}^{a_i}$ such that the projection induces birational diffeomorphism i between V and Z. In particular $V \subset \mathbf{R}^r \times \prod \mathbf{R}\mathbf{P}^{a_i}$

$$
\mathbf{RP}^r \times \prod_i \mathbf{RP}^{a_i} \hookrightarrow \mathbf{R}^{(r+1)^2 + \Sigma(a_i+1)^2}
$$

i

is a projectively closed algebraic set. Hence V is projectively closed (check details). \Box

Now assume that $L \subset W \subset \mathbb{R}^m$ be real algebraic sets, and $V \subset W \times \mathbb{R}^n$ be a projectively closed algebraic set. Let $q : \mathbb{R}^m \to$
= I Define **R** be a polynomial with $q^{-1}(0)$ $=L$. Define

$$
D_q: W \times \mathbf{R}^n \to W \times \mathbf{R}^n
$$

by $D_q(x, y) = (x, yq(x))$. D_q is a diffeomorphism on $(W - L) \times \mathbb{R}^n$ and $D_q(L)$ $\times \mathbb{R}^{n}$ = $L \times 0$. Therefore $D_q(V)$ is the quotient space of V by the equivalence relation $(x, y) \sim (x, 0)$ if $x \in L$. We call the operation $V \to D_q(V) \cup L$ (L is identified by $L \times 0$) blowing down V over L.

PROPOSITION 3.3. Given L, W, V as above, then $D_q(V) \cup L$ is an algebraic subset of $W \times \mathbb{R}^n$.

Proof: Let $p: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be an overt polynomial of degree e with V
 $p^{-1}(0)$ and let a hange above. Define a nalunamial as \mathbb{R}^m is \mathbb{R}^n . \mathbb{R}^m $p^{-1}(0)$ and let q be as above. Define a polynomial $r: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ by

$$
r(x, y) = q(x)^e p\left(x, \frac{y}{q(x)}\right)
$$

We claim $r^{-1}(0) = D_q(V) \cup L$. It is easy to see that

$$
r^{-1}(0) \cap (W - L) \times \mathbf{R}^{n} = D_q(V) \cap (W - L) \times \mathbf{R}^{n},
$$

so it suffices to show that $r^{-1}(0) \cap (L \times \mathbb{R}^n) = L \times 0$. We decompose $p(x, y)$ $= p_e(x, y) + \alpha(x, y)$ where $p_e(x, y)$ is homogeneous of degree e and $\alpha(x, y)$ is a polynomial of degree less than e. Hence if $(x, y) \in r^{-1}(0) \cap (L \times \mathbb{R}^n)$ then $r(x, y)$ 0 and $q(x) = 0$, which implies $r(x, y) = p_e(0, y) = 0$. Then $y = 0$ since p is overt, so $(x, y) \in L \times 0$. Conversely if $(x, y) \in L \times 0$ then $y = 0$ and $q(x) = 0$. Hence $r(x, y) = p_e(0, 0) = 0$, i.e. $(x, y) \in r^{-1}(0) \cap (L \times \mathbb{R}^n)$.

There is ^a more useful version of Proposition 3.3 which says that after modifying D_q we can get $D_q(V) \cup L$ to be a projectively closed algebraic set (Proposition 3.1 of $\lceil AK_6\rceil$). This allows us to iterate this blowing down process.

§4. Isolated Singularities

The topology of real algebraic sets with isolated singularities is completely understood by the following Theorem.

THEOREM 4.1 ($[AK_2]$). X is homeomorphic to an algebraic set with isolated singularities if and only if X is obtained by taking a smooth compact manifold W r with boundary $\partial W = \cup \; M_{i},\;$ where each $\; M_{i} \;$ bounds, then crushing some $i = 1$ M_i 's to points and deleting the remaining M_i 's.

