§2. A TRIPTYCH

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SOME KNOT THEORY OF COMPLEX PLANE CURVES [1]

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§1. ASPECTS OF THE "PLACEMENT PROBLEM" FOR COMPLEX PLANE CURVES

How can a complex curve be placed in a complex surface?

The question is vague; many different ways to make it more specific may be imagined. The theory of deformations of complex structure, and their associated moduli spaces, is one way. Differential geometry and function theory, curvatures and currents, could be brought in. Even the generalized Nevanlinna theory of value distribution, for analytic curves, can somehow be construed as an aspect of the "placement problem".

By "knot theory" I mean to connote those aspects of the situation that are more immediately topological. I hope to show that there is something of interest there.

§2. A TRIPTYCH

Here are three ways to interpret the phrase "knot theory of complex plane curves".

Globally: the "complex plane" is projective space $\mathbf{CP}^2$ or affine space $\mathbf{C}^2$; a "curve" is an algebraic curve (in projective space) or an algebraic or analytic curve (in affine space); here, "knot theory" has historically been largely concerned with studying the "knot group", though there are also results on "knot type".

Locally: a "complex plane curve" is the germ of a plane curve (algebraic, analytic, or formal) over $\mathbf{C}$; this is the study of singularities, and "knot theory" has been the classical knot theory of links in the 3-sphere, put to work in the service of that study.

In between: a "complex plane curve" is an analytic curve in a reasonable open set in a complex surface (chiefly, in the theory as so far developed, the

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interior of a ball or a bidisk), well-behaved at the boundary; a knot-theorist can study either of two codimension-2 situations—the complex curve in its ambient space, or the boundary of this pair.

This middle panel of the triptych has been less studied than the other two, though it is of obvious relevance to both.

§3. Résumé of basic definitions

By *complex surface* I mean a smooth manifold of 4 real dimensions, equipped with a complex structure. A *complex curve* Γ in a complex surface M is a closed subset which is locally of the form $\{(z, w) \in U \subset \mathbf{C}^2 : f(z, w) = 0\}$ where $f : U \to \mathbf{C}$ is a nonconstant complex analytic function. A *Riemann surface* is a smooth manifold of 2 real dimensions, equipped with a complex structure.

It is a fundamental fact, to which is due the especial appositeness of classical knot theory to the study of curves in surfaces, that any complex curve $\Gamma \subset M$ has a *resolution* of the following sort: There is a Riemann surface R, and a holomorphic mapping $r : R \to M$, so that $r(R) = \Gamma$; in fact, there is a discrete (possibly empty) subset $\mathscr{S}(\Gamma) \subset \Gamma$, the *singular locus of* Γ *in* M, so that the *regular locus* $\mathscr{R}(\Gamma) = \Gamma - \mathscr{S}(\Gamma)$ is a Riemann surface, and R is the union (with what turns out to be a unique topology and complex structure) of $\mathscr{R}(\Gamma)$, on which r is the identity, and a discrete set $r^{-1}(\mathscr{S}(\Gamma)) \subset R$ mapping finitely-to-one onto $\mathscr{S}(\Gamma)$.

The singular locus is, of course, exactly the set of points of Γ at which, no matter what the local representation of Γ as the zeroes of an analytic function $f(z, w)$, the (complex) gradient vector ∇f vanishes.

If P is a point of Γ, and $Q \in r^{-1}(P) \subset R$, then the germ at P of the r-image of a small disk on R centered at Q is called a *branch* of Γ at P. (Abusively, "branch" may also be used below to refer to some representatives of this germ.) Naturally, at a regular point there is only one branch; but there may be only one branch at a point, and the point still be singular.

References: [G-R], [Mi 2].

§4. Local knot theory in brief

Using local coordinates in the resolution R and the ambient surface M, one sees that each branch of a curve Γ can be parametrized either by $z = t, w = 0$ or (more interestingly) by some pair $z = t^m$, $w = t^n + c_{n+1}t^{n+1} + \ldots$