

10. Deformations of representation homomorphisms and subrepresentations

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So we have the following

- (1) $t < i + p(i)$ (by hypothesis)
- (2) $n - i \geq n + m - t - l$ or equivalently $i \leq t + l - m$
- (3) $\kappa_m + \dots + \kappa_{m-l+1} + l \leq n + m - t$
- (4) $\kappa_1 + \dots + \kappa_{p(i)} < i \leq \kappa_1 + \dots + \kappa_{p(i)+1}$.

Using (2) and (3) we have that

$$\kappa_m + \dots + \kappa_{m-l+1} \leq n - i = \kappa_1 + \dots + \kappa_m - i$$

so we have $i \leq \kappa_1 + \dots + \kappa_{m-l}$ which implies $m - l \geq p(i) + 1$ thus

$$p(i) + i \leq m - l - 1 + i \leq (m - l - 1) + (t + l - m) = t - 1$$

which contradicts (1). This proves the theorem.

9.7. *Vectorbundles and Schubert cells.* Because every positive vectorbundle over $\mathbf{P}^1(\mathbf{C})$ arises as the bundle $E(\Sigma)$ of some system Σ one has the obvious analogues of theorems 9.5 and 9.6 for positive bundles over $\mathbf{P}^1(\mathbf{C})$. Here the morphism ψ_Σ must, of course, be replaced by the classifying morphism (cf. section 3.2 above) of a positive vector bundle E , and $n + m$ and m are determined respectively as $\dim \Gamma(E, \mathbf{P}^1(\mathbf{C}))$ and $\dim E$.

10. DEFORMATIONS OF REPRESENTATION HOMOMORPHISMS AND SUBREPRESENTATIONS

10.1 *On proving Inclusion Results for Representations.* Suppose we have given a continuous family of homomorphisms of finite dimensional representations over $\mathbf{C}$ of a finite group G

$$(10.2) \quad \pi_t : M \rightarrow V$$

Suppose that $\text{Im } \pi_t \simeq \rho$ for $t \neq 0$ (and small) and that $\text{Im } \pi_0 \simeq \rho_0$. Then the representation ρ_0 is a direct summand of the representation ρ . This is seen as follows. Because the category of finite dimensional complex representations of G is semisimple there is a homomorphism of representations $\phi_0 : \text{Im } \pi_0 \rightarrow M$ such that $\pi_0 \circ \phi_0 = \text{id}$. Then $\pi_t \circ \phi_0 : \text{Im } \phi_0 \rightarrow \text{Im } \pi_t$ is still injective for small t (by the continuity of π_t) which gives us ρ_0 as a subrepresentation and hence a direct summand of ρ .

It is almost equally easy to construct a surjective homomorphism $\text{Im } \pi_t \rightarrow \text{Im } \pi_0$.

10.3. *The Inverse Result.* Inversely if ρ_0 is a subrepresentation of ρ then there is a family of representations (10.3) such that $\text{Im } \pi_t \simeq \rho$ for $t \neq 0$ and $\text{Im } \pi_0 \simeq \rho_0$, and if ρ is generated (as a $\mathbb{C}[G]$ -module) by one element one can take for M in (10.2) the regular representation. Indeed if ρ_0 is a subrepresentation of ρ then $\rho = \rho_0 \oplus \rho_1$. Let $\pi: M \rightarrow \rho = \rho_0 \oplus \rho_1$ be a surjective map of representations. Let π_0, π_1 be the two components of π . Let $s = (s_0, s_1)$ be a section of π . Then $\pi_0 s_0 = \text{id}, \pi_1 s_1 = \text{id}, \pi_0 s_1 = 0, \pi_1 s_0 = 0$ and it follows that $\pi(t)$ consisting of the components π_0 and $t\pi_1$ is still surjective. Hence $\text{Im } \pi(t) = \rho$ and $\text{Im } \pi(0) = \rho_0$.

11. A FAMILY OF REPRESENTATIONS OF S_{n+m} PARAMETRIZED BY $G_n(\mathbb{C}^{n+m})$

11.1. *Construction of the Family.* Let M be the regular representation of S_{n+m} . That is M has a basis e_σ , $\sigma \in S_{n+m}$ and S_{n+m} acts on M by the formula $\tau(e_\sigma) = e_{\tau\sigma}$, for all $\tau \in S_{n+m}$. Now consider the universal bundle ξ_m over $G(\mathbb{C}^{n+m})$ and the $n + m$ holomorphic section $\varepsilon_1, \dots, \varepsilon_{n+m}$ defined by

$$\varepsilon_i(x) = e_i \bmod x \in \mathbb{C}^{n+m}/x,$$

where e_i is the i -th standard basis vector. Take the $(m+n)$ -fold tensor product of ξ_m and define a family of homomorphisms parametrized by $G_n(\mathbb{C}^{n+m})$ by

$$(11.2) \quad \pi_x: M \rightarrow \xi_m(x)^{\otimes(n+m)}, e_{\sigma^{-1}} \mapsto \varepsilon_{\sigma(1)}(x) \otimes \dots \otimes \varepsilon_{\sigma(n)}(x)$$

More precisely (11.2) defines a homomorphism of vectorbundles

$$(11.3) \quad G_n(\mathbb{C}^{n+m}) \times M \rightarrow \xi_m^{\otimes(n+m)}$$

The group S_{n+m} acts on $\xi_m(x)^{\otimes(n+m)}$ by permuting the factors and it is a routine exercise to see that π_x is equivariant with respect to this action, i.e. that $\pi_x(\tau v) = \tau \pi_x(v)$ for all $v \in M, \tau \in S_{n+m}$. (Here the product $\tau\sigma \in S_{n+m}$ is interpreted as first the automorphism σ of $1, \dots, n+m$ and then the automorphism τ .)

Thus $\text{Im } \pi_x = \pi(x)$ is a representation of S_{n+m} for all x giving us a family of representations parametrized by $G_n(\mathbb{C}^{n+m})$. Fixing a point $x_0 \in G_n(\mathbb{C}^{n+m})$ and choosing m independent sections of ξ_m in a neighbourhood U of x_0 , this gives us families of homomorphisms of representations