4. Schubert Cells

the point of $\mathbf{G}_n(\mathbf{C}^{n+m})$ represented by $\mathrm{Ker}(V \to E(x))$. This gives a holomorphic map $\Psi_E : M \to \mathbf{G}_n(\mathbf{C}^{n+m})$ such that the pullback of ξ_m by means of Ψ_E is isomorphic to E, $\Psi_E^! \xi_m \simeq E$. It is universality properties such as this one which account for the importance of the bundles ξ_m and η_n in differential and algebraic topology [16], algebraic geometry and also system and control theory (cf. [22, 23] and the references therein for the last mentioned).

The bundle ξ_m has a number of obvious holomorphic sections, viz. the sections defined by $\varepsilon_i(x) = e_i \bmod x$ where e_i is the i-th standard basis vector of $\mathbf{C}^{n+m}$, $i = 1, \ldots, n+m$. And, as a matter of fact, it is not difficult to show that $\Gamma(\xi_m, \mathbf{G}_n(\mathbf{C}^{n+m}))$ is $(n+m)$-dimensional and that the $\varepsilon_1, \ldots, \varepsilon_{n+m}$ form a basis for this space of holomorphic sections; cf. subsection 8.1 below.

4. Schubert Cells

4.1. *Schubert Cells.* Consider again the Grassmann manifold $\mathbf{G}_n(\mathbf{C}^{m+n})$. Let $\underline{A} = (A_1, \ldots, A_n)$ be a sequence of n-subspaces of $\mathbf{C}^{n+m}$ such that $0 \neq A_1 \subset A_2 \subset \ldots \subset A_n$ with each containment strict. To each such sequence $\underline{A}$ we associate the closed subset

$$SC(\underline{A}) = \{x \in \mathbf{G}_n(\mathbf{C}^{m+n}) \mid \dim(x \cap A_i) \geqslant i\} \tag{4.2}$$

and call it the closed Schubert-cell of the sequence $\underline{A}$. In particular if

$$0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n \leqslant n + m$$

is a strictly increasing sequence of natural numbers less than or equal to $n + m$ then we define (setting $\gamma = (\gamma_1, \ldots, \gamma_n)$)

$$SC(\gamma) = SC(\mathbf{C}^{\gamma_1}, \ldots, \mathbf{C}^{\gamma_n}) \tag{4.3}$$

where $\mathbf{C}^r$ is viewed as the subspace of all vectors in $\mathbf{C}^{n+m}$ whose last $n + m - r$ coordinates are zero.

4.4 *Flag Manifolds and the Bruhat Decomposition.* A *flag* in $\mathbf{C}^{n+m}$ is a sequence of subspaces $\underline{F} = F_1 \subset \ldots \subset F_{n+m} \subset \mathbf{C}^{n+m}$ such that $\dim F_i = i$. Let $Fl(\mathbf{C}^{n+m})$ denote the analytic manifold of all flags in $\mathbf{C}^{n+m}$. There is a natural holomorphic mapping $Fl(\mathbf{C}^{n+m}) \to \mathbf{G}_n(\mathbf{C}^{n+m})$ given by associating to a flag $\underline{F}$ its n-th element F_n. The flag manifold can be seen as the space of all cosets Bg, $g \in \mathbf{GL}_{n+m}(\mathbf{C})$ where B is the Borel subgroup of all lower triangular matrices in $\mathbf{GL}_{n+m}(\mathbf{C})$. The mapping $\mathbf{GL}_{n+m}(\mathbf{C}) \to Fl(\mathbf{C}^{n+m})$ associates to a matrix g the flag $\underline{F}(g)$ whose i-th element is the subspace of $\mathbf{C}^{n+m}$ spanned by the first i row vectors of g.

Now view S_{n+m}, the symmetric group on $n + m$ letters as a subgroup of $\mathbf{GL}_{n+m}(\mathbf{C})$ by letting it permute the basis vectors $(\sigma(e_i) = e_{\sigma(i)})$. Then in $\mathbf{GL}_{n+m}(\mathbf{C})$ we have the so-called Bruhat decomposition.

$$\mathbf{GL}_{n+m}(\mathbf{C}) = \bigcup_{\sigma} B\,\sigma\,B \qquad \text{(disjoint union)} \tag{4.5}$$

Where σ runs through the Weyl group S_{n+m} of $\mathbf{GL}_{n+m}(\mathbf{C})$. An analogous decomposition holds in a considerably more general setting (reductive groups, cf. [24], section 28).

4.6. *The Bruhat order (also sometimes called Bernstein-Gelfand-Gelfand, or BGG order).* The closure of a double coset $B\,\sigma\,B$ is necessarily a union of other double cosets (by continuity). This defines an ordering on the Weyl group S_{n+m} defined by

$$\sigma > \tau \leftrightarrow \overline{B\,\sigma\,B} \supset B\,\tau\,B \tag{4.7}$$

This ordering plays a considerable role in the study of cohomology of flag spaces [1] and also in the theory of highest weight representations [25, 26].

Let H be the subgroup of $\mathbf{G}_{n+m}(\mathbf{C})$ consisting of all block lower triangular matrices of the form $\begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$, $S_{11} \in \mathbf{GL}_n(\mathbf{C})$, $S_{22} \in \mathbf{GL}_m(\mathbf{C})$, S_{21} an arbitrary $m \times n$ matrix. Then, using the remarks made in subsection 4.4 above, one sees that $\mathbf{G}_n(\mathbf{C}^{n+m})$ is the coset space $\{Hg \mid g \in \mathbf{GL}_{n+m}(\mathbf{C})\}$. Now let $\sigma \in S_{n+m}$ and let $\gamma_1 < \dots < \gamma_n$ be the n natural numbers in increasing order determined by

$$\sigma(e_{\gamma_i}) \in \{e_1, \dots, e_n\},\ i = 1, \dots, n\,.$$

Then one easily sees that the image of $B\,\sigma\,B$ under $\mathbf{GL}_{n+m}(\mathbf{C}) \to \mathbf{G}_n(\mathbf{C}^{n+m})$, i.e. the set of all spaces spanned by matrices of the form $h\,\sigma\,b$, $h \in H$, $b \in B$, is the open Schubert cell of all elements in $\mathbf{G}_n(\mathbf{C}^{n+m})$ spanned by the rows of a matrix of the form

$$\begin{array}{ccccc}
* \dots * & 0 \dots 0 & 0 & \dots & 0 \\
* \dots * & * \dots * & 0 & \dots & 0 \\
 & & & & \\
* \dots * & * \dots * & * \dots & * & 0 \dots 0 \\
\uparrow & \nwarrow & & & \\
\text{column } \gamma_1 & \text{column } \gamma_2 & & &
\end{array}$$

where the last * in each row is nonzero. The closure of this open Schubert-cell is the Schubert-cell $SC(\gamma)$ defined in (4.3) above.

One easily checks that

$$SC(\mu) \subset SC(\gamma) \leftrightarrow \mu_i \leqslant \gamma_i, \; i = 1, \ldots, n \tag{4.8}$$

and this order on the Schubert cells $SC(\gamma)$, or the equivalent ordering on n-tuples of natural numbers, is therefore a quotient of the Bruhat order on the Weyl group S_{n+m}. It is the induced order on the set of cosets $(S_n \times S_m)\sigma$, $\sigma \in S_{n+m}$. (Obviously if $\tau \in S_n \times S_m$, then $\tau\sigma(e_{\gamma_i}) \in \{e_1, \ldots, e_n\}$ if $\sigma(e_{\gamma_i}) \in \{e_1, \ldots, e_n\}$.) (And inversely the Bruhat order is determined by the associated orders of Schubert cells in the sense that $\sigma > \tau$ in S_n iff for all $k = 1, \ldots, n-1$ we have for the associated Schubert cells in $\mathbf{G}_k(\mathbf{C}^n)$ that $SC(\sigma) \supset SC(\tau)$; this is a rather efficient way of calculating the Bruhat order on the Weyl group S_n.)

5. Interrelations

Now that we have defined the concepts we need we can start to describe some interrelations between the various manifestations of the specialization order we discussed in section 2 above.

5.1. *Overview of the Various Relations.* A schematic overview of the various interconnections is given by the following diagram. In this diagram we