10. Another fruit of the Perron tree. A PROBLEM ON DOUBLE FOURIER SERIES

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the sets become smaller and smaller, the mean density approaches a number, the density at the point. The Nikodym set shows in an easy way that one has to be very careful at choosing *reasonable sets.* As Zygmund observed (see the end of Nikodym's paper in 1927), it follows from the Nikodym set that if we take something apparently so reasonable as the system of all rectangles centered at the corresponding points, the mean densities can diverge. This, however, does not happen if the system is that of all circles or squares containing the points. Considerations of this type have given rise to the modern theory of differentiation of integrals.

10. ANOTHER FRUIT OF THE PERRON TREE. A PROBLEM ON DOUBLE FOURIER SERIES

A famous problem in Fourier analysis, open for a long time, has been recently solved in a rather simple way be the use of the Perron tree.

For a periodic function of two variables $f(x, y)$ of period 1 in each variable one can define its Fourier coefficients setting for $m = 0, \pm 1, \pm 2, \ldots, n = 0, \pm 1, \pm 2, \ldots$

$$a_{mn} = \int_0^1 \int_0^1 f(x, y) e^{-2\pi imx} e^{-2\pi iny}\, dx\, dy$$

and one can construct the corresponding Fourier series

$$\sum_{m,n} a_{mn}\, e^{2\pi imx}\, e^{2\pi iny}.$$

One can consider the partial sums of this series in several ways, in order to explore whether they converge or not to the original function. Thus, for example, one can consider the "square" sums

$$S_P f(x, y) = \sum_{\substack{|m| < P \\ |n| < P}} a_{mn}\, e^{2\pi imx}\, e^{2\pi iny}$$

or else the "rectangular" sums

$$S_{M,N} f(x, y) = \sum_{\substack{|m| < M \\ |n| < N}} a_{mn}\, e^{2\pi imx}\, e^{2\pi iny}$$

and examine whether in some sense $S_P f \to f$ as $P \to \infty$ or $S_{M,N} f \to f$ as $M, N \to \infty$. One can also consider the "circular" sums

$$S^R f(x, y) = \sum_{m^2 + n^2 \leqslant R^2} a_{mn}\, e^{2\pi imx}\, e^{2\pi iny}$$

As a consequence of the construction of the Perron tree one can prove, for example, that there are functions $f \in L^p$, $1 < p < 2$, such that $S^R f(x, y)$ diverges at almost each point (x, y). For this result we refer to the papers by C. Fefferman in 1970 and 1971.

Our short excursion comes to confirm what happens so often in Mathematics. Apparently idle and superfluous questions give rise to very interesting and important portions of mathematics, useful in many respects. As Littlewood used to say, a good mathematical game is worth many theorems.

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