

## 6. The solution of the needle problem

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## 6. THE SOLUTION OF THE NEEDLE PROBLEM

The Perron tree gives a simple solution to the Kakeya problem. First we shall show how a needle can go from a straight line to another one parallel to it covering an arbitrarily small area. Let us observe Figure 12.

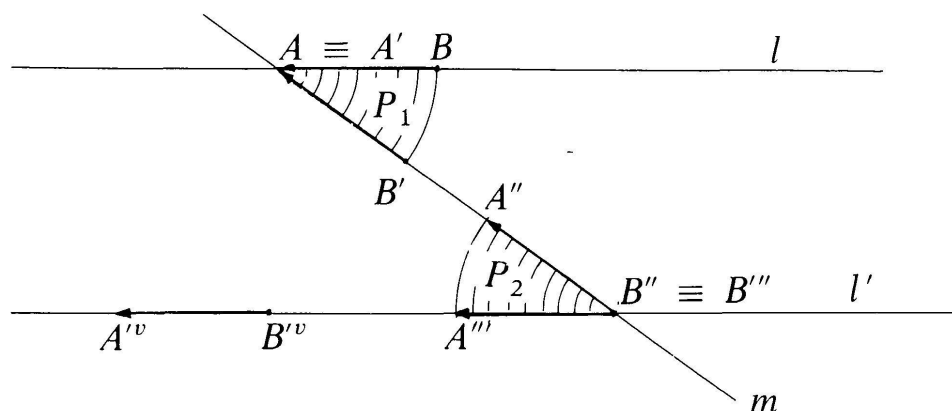


FIGURE 12

If the needle  $AB$  is on  $l$  and we wish to translate it to  $l'$ , we draw through  $A$  a straight line  $m$  intersecting  $l$  and  $l'$  whose direction can be as close to that of  $l$  and  $l'$  as we wish. From  $AB$  we move to  $A^I B^I$  covering area  $P_1$ , from  $A^I B^I$  to  $A^{II} B^{II}$  covering null area, from  $A^{II} B^{II}$  to  $A^{III} B^{III}$  covering  $P_2$ . Now  $P_1 + P_2$  can be made arbitrarily small if the slope of  $m$  over  $l$  and  $l'$  is small. From  $A^{III} B^{III}$  we can move to any other position  $A^{IV} B^{IV}$  on  $l'$  covering again null area.

Let us now assume that the needle is on the side  $AB$  of the initial triangle  $ABC$ . We can assume that  $ABC$  is an equilateral triangle and that its height is of the same length as that of the needle. Let us see how we can move the needle to  $AC$  sweeping an area smaller than  $\eta/3$  with a positive  $\eta$  arbitrarily small.

We construct a Perron tree  $P$  starting from  $ABC$  with an  $\varepsilon > 0$  such that  $\varepsilon S(ABC) < \eta/6$ . Here, as before,  $S(ABC)$  denotes the area of the triangle  $ABC$ . Let  $n$  be the number of small triangles  $T_1, T_2, \dots, T_n$  in which we have to divide  $ABC$  and let  $T'_1 \equiv T_1, T'_2, \dots, T'_n$  be their corresponding final positions in the Perron tree. We shall move the needle inside  $P$  and inside  $n$  figures like that of Figure 12 with an area  $J$  each one such that  $nJ < \eta/6$ . If the needle is on  $AB$  with an extremity on  $A$ , it can move inside  $T'_1 \equiv T_1$ , therefore inside  $P$ , until it comes over the right hand side of  $T'_1$ . Now  $T'_2$  has its left hand side parallel to the right

hand side of  $T'_1$ . Therefore it can move, using the above construction, covering an area  $J$ . Within  $T'_2$ , and so within  $P$ , it can move to the right hand side of  $T'_2$ . From there to the left hand side of  $T'_3$  and so on until it comes to  $AC$ , covering area less than  $\eta/3$ .

It is clear that with three equilateral triangles and three repetitions of this process we can turn the needle around covering area smaller than  $\eta$ .

## 7. THE CONSTRUCTION OF THE BESICOVITCH SET

The Besicovitch set is also easily built starting from the Perron tree by means of the following auxiliary construction :

(\*\*) *Given an arbitrary parallelogram  $ABCD$  and  $\varepsilon > 0$ , it is possible to construct a finite number of closed parallelograms  $\omega_1, \omega_2, \dots, \omega_n$  so that (see Fig. 13):*

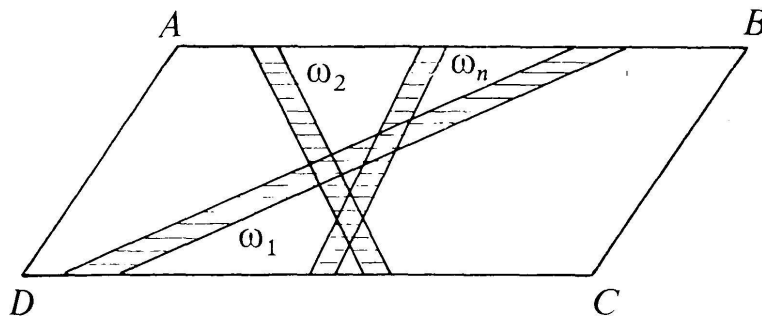


FIGURE 13

- (a) *Each one has one basis on  $AB$  and another one on  $CD$ .*
- (b) *The area of their union is less than  $\varepsilon$ .*
- (c) *For each segment joining a point of  $AB$  to another one of  $CD$  there exists inside some  $\omega_j$  a segment parallel to it of the same length.*

To see this, given  $ABCD$  and  $\varepsilon > 0$  we first take two strips  $\omega_1$  and  $\omega_2$  as indicated in Figure 14 such that  $S(\omega_1) + S(\omega_2) < \varepsilon/4$ . We take now a point  $L$  of  $UV$  so that  $LC$  is parallel to  $UT$ . Then we divide  $VC$  into intervals with the same length smaller than that of  $DV$  and we join  $L$  to the extreme points of these intervals. A typical triangle of the ones so obtained is  $LMN$ . Let  $p$  be