6. Unitarizability of irreducible subrepresentations OF THE PRINCIPAL SERIES

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

theorem for noncompact semisimple Lie groups due to Casselman (cf. WALLACH [47, Cor. 7.5]). Casselman's theorem improves HARISH-CHANDRA'S [22, Theorem 4] *subquotient theorem.*

5.6.3. The generalized Abel transform $f \to F_f^\delta$ can be defined for general K-type δ. It was introduced by HARISH-CHANDRA [24, p. 595] in the spherical case, TAKAHASHI [40, §2] in the case $G = SO_0(n, 1)$ and WARNER [49, 6.2.2] in the general case. The injectivity of this transform holds generally, cf. WARNER [49]. The image of $I_{c,\delta}^\infty(G)$ under this transform is known in the spherical case (cf. GANGOLLI [16]) and if G has real rank 1 and δ is one-dimensional (cf. WALLACH [46]), but seems to be unknown in the general case (cf. WARNER [49, p. 36]).

5.6.4. In [39] TAKAHASHI also reduces the proof of Theorem 5.4 to Proposition 5.5. However, he proves Prop. 5.5 by considering eigenfunctions of the Casimir operator, since he did not know, then, how to invert the transform $f \to F_f^n$. In [42] he independently obtained a proof of Prop. 5.5 similar to ours. Earlier, in [40, §4.1] he used a similar method in the spherical case of $G = SO_0(n, 1)$. NAIMARK [34, Ch. 3, §9] proved the subquotient theorem for $SL(2, \mathbf{C})$ by methods somewhat related to ours.

5.6.5. Part of Lemma 5.8 is contained in WHITNEY [50]. See SCHWARZ [37] for a theorem on C^∞-functions which are invariant under a more general Weyl group.

5.6.6. Theorem 5.10 more generally holds with Gegenbauer polynomials of integer of half integer order as kernels, cf. DEANS [6], [7], KOORNWINDER [27, §5.9]. Deans' proof uses the inversion formula for the Radon transform. The author's proof uses Weyl fractional integral transforms and generalized fractional integral transforms studied by SPRINKHUIZEN [38]. MATSUSHITA [30, §2.3] considers the transformation $f \to F_f^n$ for general real n in the context of the universal covering group of $SL(2, \mathbf{R})$ and he derives the inversion formula with a proof due to T. Shintani, which uses Mellin transforms.

6. UNITARIZABILITY OF IRREDUCIBLE SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

6.1. A CRITERIUM FOR UNITARIZABILITY

Remember that a representation of an lcsc. group G on a Hilbert space is strongly continuous if and only if it is weakly continuous (cf. WARNER [48, Prop. 4.2.2.1]). Thus, if τ is a (strongly continuous) Hilbert representation of G then $\tilde{\tau}$ defined by

(6.1) $$\tilde{\tau}(g) := \tau(g^{-1})^*, \; g \in G\,,$$

is again a (strongly continuous) Hilbert representation of G on $\mathscr{H}(\tau)$. The representation $\tilde{\tau}$ is called the *conjugate contragredient* to τ. The representation τ is unitary iff $\tilde{\tau} = \tau$.

THEOREM 6.1. *Let G be an lcsc. group with compact abelian subgroup K. Let τ be a K-multiplicity free representation of G. Let $\{\phi_\delta\}$ be a K-basis for $\mathscr{H}(\tau)$. Let $c_\delta(\delta \in \mathscr{M}(\tau))$ be positive real numbers. Then the following statements are equivalent to each other:*

(a) *τ is Naimark equivalent to some unitary representation.*

(b) $\tau \overset{A}{\simeq} \tilde{\tau}$ *with* $A\phi_\delta = c_\delta \phi_\delta(\delta \in \mathscr{M}(\tau))$.

(c) $\overline{\tau_{\gamma,\delta}(g^{-1})} = \dfrac{c_\delta}{c_\gamma}\, \tau_{\delta,\gamma}(g),\ \gamma,\ \delta \in \mathscr{M}(\tau),\ g \in G.$

If, moreover, τ is irreducible then (a), (b) *and* (c) *are equivalent to:*

(d) *For some* $\delta \in \mathscr{M}(\tau)$ *we have*

$$\overline{\tau_{\gamma,\delta}(g^{-1})} = \frac{c_\delta}{c_\gamma}\, \tau_{\delta,\gamma}(g),\ g \in G,\ \textit{for all}\ \ \gamma \in \mathscr{M}(\tau)\,.$$

If (b) *holds then* $\tau(g)(g \in G)$ *is unitary with respect to a new inner product* $<\cdot,\cdot>$ *on* $\mathscr{D}(A)$ *defined by*

(6.2) $$<\phi_\gamma, \phi_\delta> := \begin{cases} 0 & \text{if } \gamma \neq \delta, \\ c_\delta & \text{if } \gamma = \delta. \end{cases}$$

Proof. First observe that $\tilde{\tau}(k)\phi_\delta = \delta(k)\phi_\delta(k \in K)$, so $\{\phi_\delta\}$ is a K-basis with respect to $\tilde{\tau}$ as well. We have

(6.3) $$\tilde{\tau}_{\gamma,\delta}(g) = \overline{\tau_{\delta,\gamma}(g^{-1})}\,.$$

<u>(a) ⇒ (b)</u> : Let $\tau \overset{B}{\simeq} \sigma$ with σ unitary. Then $\sigma = \tilde{\sigma}$ and $\tilde{\sigma} \overset{B^*}{\simeq} \tilde{\tau}$. Let $\{\psi_\delta\}$ be a K-basis for $\mathscr{H}(\sigma)$. Let $B\phi_\delta = b_\delta \psi_\delta(\delta \in \mathscr{M}(\tau))$. Then, by Theorem 4.5:

$$\tilde{\tau}_{\gamma,\delta} = \frac{\overline{b_\gamma}}{\overline{b_\delta}}\, \sigma_{\gamma,\delta} = \left|\frac{b_\gamma}{b_\delta}\right|^2 \tau_{\gamma,\delta}\,,$$

so (b) holds.

(b) ⇒ (a) : Assume (b). Then A is self-adjoint and positive definite. Define a new inner product $<\cdot,\cdot>$ on $\mathscr{D}(A)$ by $<v, w> := (Av, w)$. Then, for $v, w \in \mathscr{D}(A)$, $g \in G$, we have:

$$\begin{aligned} <\tau(g)v, \tau(g)w> &= \left(A\tau(g)v, \tau(g)w\right) = \left(\tilde{\tau}(g^{-1})A\tau(g)v, w\right) \\ &= \left(A\tau(g^{-1})\tau(g)v, w\right) = (Av, w) = <v, w> , \end{aligned}$$

i.e. $<\tau(g)v, \tau(g)w> = <v, w>$. Thus τ is a unitary representation on $\mathscr{D}(A)$ with respect to the new inner product. (Weak continuity of τ is easily proved.) Let σ be the extension of this representation to a unitary representation in the Hilbert space completion $\mathscr{H}(\sigma)$ of $\mathscr{D}(A)$ with respect to $<\cdot,\cdot>$. Then $\tau \overset{B}{\simeq} \sigma$, where B is the closure of the identity operator on $\mathscr{D}(A)$ (cf. Lemma 4.4). Note that we have also proved the last part of the theorem.

The equivalence of (c) or (d) with (b) follows from Theorem 4.5. □

6.2. THE CASE $SU(1, 1)$

It follows from (2.30) that

(6.4) $$\overline{c_{\xi,\lambda,n,m}} = (-1)^{m-n}\, c_{\xi,-\bar{\lambda},m,n}\,.$$

Combination of (6.3), (2.29) and (6.4) yields

(6.5) $$\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\bar{\lambda}}\,.$$

In §6.1 we showed that a necessary condition for unitarizability of an irreducible subquotient representation τ of $\pi_{\xi,\lambda}$ is the equivalence of τ and $\tilde{\tau}$. In view of (6.5) and Theorem 4.7 this is only possible if $\bar{\lambda} = \pm\lambda$, that is, if λ is real or imaginary. If λ is imaginary then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,\lambda}$, so $\pi_{\xi,\lambda}$ is already unitary. Let us now examine the case that λ is real and nonzero. Then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\lambda}$. If τ is an irreducible subquotient representation of $\pi_{\xi,\lambda}$ then $\tau \overset{A}{\simeq} \tilde{\tau}$ with (cf. (4.10))

(6.6) $$A\phi_m = c_{\xi,\lambda,m}\,\phi_m,\ \phi_m \in \mathscr{H}(\tau)\,,$$

where $c_{\xi,\lambda,m}$ is given by (4.9). Now a sufficient condition for the unitarizability of τ is that the coefficients $c_{\xi,\lambda,m}$ are all positive or all negative for $\phi_m \in \mathscr{H}(\tau)$. Referring to the classification in Theorem 3.4 we will examine these coefficients. (Because of equivalence, it is not necessary to treat the cases where $\lambda < 0$.)

(a) $\pi_{0,\lambda}(\lambda>0, \lambda\notin\mathbf{Z}+\frac{1}{2})$.

$$c_{0,\lambda,m} = \frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}}, \quad m\in\mathbf{Z}.$$

$c_{0,\lambda,m}$ has fixed sign iff $0 < \lambda < \frac{1}{2}$.

(b) $\pi_{\frac{1}{2},\lambda}(\lambda>0, \lambda\notin\mathbf{Z})$.

$$c_{\frac{1}{2},\lambda,m} = \frac{(-\lambda)_{m+\frac{1}{2}}}{(\lambda)_{m+\frac{1}{2}}}, \quad m + \tfrac{1}{2}\in\{0, 1, 2, \ldots\}.$$

No fixed sign.

(c) $\pi^{+}_{\xi,\lambda}$ and $\pi^{-}_{\xi,\lambda}(\lambda+\xi\in\mathbf{Z}+\frac{1}{2}, \lambda>0)$.

$$c_{\xi,\lambda,m} = \frac{(|m| - (\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}}, \quad m\in\mathbf{Z} + \xi, |m| \geqslant \lambda + \tfrac{1}{2}.$$

Fixed sign.

(d) $\pi^{0}_{\xi,\lambda}(\lambda+\xi\in\mathbf{Z}+\frac{1}{2}, \lambda>0)$.

$$c_{\xi,\lambda,m} = \frac{(-1)^{m-\xi}}{(\lambda-\frac{1}{2}+m)!(\lambda+\frac{1}{2}-m)!}, \quad m\in\left\{-\lambda + \tfrac{1}{2}, -\lambda + \tfrac{3}{2}, \ldots, \lambda - \tfrac{1}{2}\right\}.$$

No fixed sign except if $\lambda = \frac{1}{2}, \xi = 0$.

Combining these results with Theorems 3.4, 4.7 and 5.4 and Prop. 4.2 we reobtain BARGMANN'S [2] classiffication of all irreducible unitary representations of $SU(1, 1)$:

THEOREM 6.2. *Any irreducible unitary representation of* $SU(1,1)$ *is unitarily equivalent to one and only one of the following representations:*

1) $\pi_{\xi,i\nu}(\xi=0, \frac{1}{2}, \nu>0)$, $\pi_{0,0}$, $\pi^{+}_{\frac{1}{2},0}$, $\pi^{-}_{\frac{1}{2},0}$ *(unitary principal series).*

2) $\pi_{0,\lambda}(0<\lambda<\frac{1}{2})$ on Cl Span$\{\ldots, \phi_{-1}, \phi_0, \phi_1, \ldots\}$

with respect to the inner product

$$< \phi_m \, \phi_n > := \frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}} \delta_{m,n} \textit{ (complementary series).}$$

3) $\pi^{+}_{\xi,\lambda}$ *and* $\pi^{-}_{\xi,\lambda}\left(\xi = 0 \text{ or } \frac{1}{2}, \lambda = \xi + \frac{1}{2}, \xi + \frac{3}{2}, \ldots\right)$

on

$$Cl\ \mathrm{Span}\{\phi_{\lambda+\frac{1}{2}}, \phi_{\lambda+3/2}, \ldots\}$$

and

$$Cl\ \mathrm{Span}\{\ldots, \phi_{-\lambda-3/2}, \phi_{-\lambda-\frac{1}{2}}\},$$

respectively, with respect to the inner product

$$<\phi_m, \phi_n> := \frac{(|m| - (\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}} \delta_{m,n} \text{ (discrete series).}$$

4) $\pi^0_{0,\frac{1}{2}}$ *(identity representation).*

6.3. Notes

6.3.1. Following BARGMANN [2], most authors prove Theorem 6.2 by infinitesimal methods. VILENKIN [43, Ch. VI] uses the method of the present paper. TAKAHASHI [39, §6] decides about unitarizability by considering whether $\pi_{\xi,\lambda,n,n}$ is a positive definite function on G.

6.3.2. A method related to this section was used in FLENSTED-JENSEN & KOORNWINDER [15] in order to find all irreducible unitary spherical representations of non-compact semisimple Lie groups G of rank one. They examined the nonnegativity of the coefficients in the addition formula for the spherical functions on G. See also [27, §6.4].

6.3.3. A generalization of Theorem 6.1 can be formulated for not necessarily abelian K and, partly, for K-finite τ, cf. [27, Theorems 6.4, 6.5].