5. The line-sphere transformation

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

5. THE LINE-SPHERE TRANSFORMATION

The homogeneous contact manifold of co-directions in complex projective space P^3, obtained from the simple complex Lie algebra of type A_3, must coinside with that of oriented co-directions in complex Euclidean space E^3, obtained from the algebra of type D_3, in view of the isomorphisms $A_3 \simeq D_3$. To exhibit this explicitly, we introduce a third homogeneous contact manifold in terms of which both of these can be conveniently described, namely, the space of lines in the quadric Ω^4 in P^5 of Section 1.

5.1. We carry out the construction of 2.10 for the simple complex Lie algebras of type B_l and D_l, making the restriction to type D_3 later.

Let $\mathfrak{g} = \mathfrak{o}\,(A; \mathbf{C})$, complex square matrices X for which ${}^tXA + AX = 0$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & 1_l & 0 \end{bmatrix} \quad \text{in case } B_l$$

or

$$A = \begin{bmatrix} 0 & 1_l \\ 1_l & 0 \end{bmatrix} \quad \text{in case } D_l \quad ,$$

that is, the quadratic form defining $\mathfrak{g}$ is

$$\xi_0{}^2 + 2\xi_1\xi_{l+1} + \ldots + 2\xi_l\,\xi_{2l}$$

or

$$2\xi_1\xi_{l+1} + \ldots + 2\xi_l\xi_{2l}$$

respectively [4, (16.3) and (16.4)].

We exhibit the details of the construction for the case of D_l. For B_l one need only carry along an additional initial row and column in the matrices, as well as the corresponding roots; the conclusions are the same.

Thus $\mathfrak{g}$ consists of $2l$ by $2l$ matrices of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{bmatrix} ,$$

where X_1 is l by l and arbitrary and X_2 and X_3 are l by l and skew-symmetric. For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take diagonal matrices H of the form

$$H = \operatorname{diag}(h_1, \dots, h_l \mid -h_1, \dots, -h_l).$$

Let δ_i, $i = 1, 2, \dots, l$ be the linear function on $\mathfrak{h}$ which assigns h_i to H: $\delta_i(H) = h_i$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are

$$\pm\delta_i \pm \delta_j \qquad i, j = 1, 2, \dots, l \quad \text{and} \quad i \neq j$$

and the root vector E_α corresponding to the root α is

$$E_{\delta_i - \delta_j} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix}, \quad i \neq j,$$

$$E_{\delta_i + \delta_j} = \begin{bmatrix} 0 & E_{ij} - E_{ji} \\ 0 & 0 \end{bmatrix}, \quad i < j,$$

$$E_{-\delta_i - \delta_j} = \begin{bmatrix} 0 & 0 \\ E_{ji} - E_{ij} & 0 \end{bmatrix}, \quad i < j,$$

where E_{ij} is the l by l matrix with 1 in the i^{th} row and j^{th} column and 0s elsewhere [4, (16.3)]. A system of simple roots is

$$\delta_1 - \delta_2, \ \delta_2 - \delta_3, \dots, \delta_{l-1} - \delta_l, \quad \text{and} \quad -\delta_1 - \delta_2,$$

(this is not the same choice as in [4, (16.3)]), for which the maximal root is

$$\rho = -\delta_{l-1} - \delta_l,$$

[4, App., Table E]. The Killing form of $\mathfrak{g}$ is $\langle X, Y \rangle = (2l-2)\operatorname{tr}(XY)$, but we replace this with $\langle X, Y \rangle = \frac{1}{2}\operatorname{tr}(XY)$ for convenience. Then the H_α in $\mathfrak{h}$ are given by

$$H_{\pm\delta_i \pm \delta_j} = \operatorname{diag}(0, \dots, 0, \pm 1, 0, \dots, 0, \pm 1, 0, \dots, 0 \mid \text{———}),$$

where the ± 1s occur in the i^{th} and j^{th} entries and the second l entries are the negatives of the first l entries. Especially,

$$H_\rho = \operatorname{diag}(0, \dots, 0, -1, -1 \mid 0, \dots, 0, 1, 1).$$

It is now straightforward to determine for which roots α we have $\langle H_\rho, H_\alpha \rangle \geqslant 0$ and find that $\mathfrak{p}$ in (i) of 2.9 consists of matrices of the form

$$\left[\begin{array}{c|c}
\begin{matrix}\text{negative}\\ \text{transpose}\\ \text{of lower}\\ \text{right}\end{matrix} &
\begin{matrix}
\text{arbitrary} & & 0 & 0\\
(l-2)\text{ by }(l-2) & & \vdots & \vdots\\
\text{skew-} & & & \\
\text{symmetric} & & 0 & 0\\
0 \;\cdots & 0 & 0 & 0\\
0 \;\cdots & 0 & 0 & 0
\end{matrix}\\
\hline
\begin{matrix}\text{arbitrary}\\ l\text{ by }l\\ \text{skew-}\\ \text{symmetric}\end{matrix} &
\begin{matrix}
* \;\cdots & * & 0 & 0\\
\vdots & \vdots & \vdots & \vdots\\
 & & 0 & 0\\
 & & * & *\\
* \;\cdots & * & * & *
\end{matrix}
\end{array}\right],$$

where the starred entries are arbitrary.

5.2. The connected centerless simple group $G = PSO\,(A;\,\mathbf{C})$ is transitive on the lines of the quadric Ω^{2l-2}

$$\xi_1\xi_{l+1} + \dots + \xi_l\xi_{2l} = 0$$

in P^{2l-1} by Witt's theorem. The Lie algebra of the isotropy subgroup of the line l_0 joining

$${}^t(0, \dots, 0, 1, 0) \quad \text{and} \quad {}^t(0, \dots, 0, 0, 1)$$

is $\mathfrak{p}$. Hence

$$G/P = \text{space of lines in } \Omega^{2l-2}.$$

The element $W = E_\rho$ of $\mathfrak{p}$ giving the contact structure on G/P, as in 2.7, is

$$W = \left[\begin{array}{c|c}
0 & 0\\
\hline
\begin{matrix} 0 \\ \\ \\ \begin{matrix}0 & -1\\ 1 & 0\end{matrix}\end{matrix} & 0
\end{array}\right]$$

In general, the construction of 2.10 gives the $(2n-1)$-dimensional homogeneous contact manifold of lines in the quadric Ω^{n+1} in P^{n+2}, where Ω^{n+1} is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l} = 0$$

in case B_l when n is even, $n+3 = 2l+1$, and Ω^{n+1} is Ω^{2l-2} above in case D_l when n is odd, $n+3 = 2l$; $n \geqslant 2$.

The real contact structure on the $(2n-1)$ dimensional space of lines of Ω^{n+1} in real projective space P^{n+2} is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the one- or two- component centerless group $PSO(A; \mathbf{R})$ consisting of real contact automorphisms.

5.3. The line joining $x = {}^t(x_0, x_1, x_2, x_3)$ and $y = {}^t(y_0, y_1, y_2, y_3)$ in complex projective space P^3 has Plücker coordinates $p_{ij} = x_iy_j - x_jy_i$. These coordinates are the coefficients of the bivector $x \wedge y$ with respect to the basis

$$e_1 \wedge e_2\,,\; e_3 \wedge e_1\,,\; e_2 \wedge e_3\,,\; e_0 \wedge e_3\,,\; e_0 \wedge e_2\,,\; e_0 \wedge e_1\,,$$

where $e_0 = {}^t(1, 0, 0, 0), \dots, e_3 = {}^t(0, 0, 0, 1)$, and satisfy

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0\,,$$

[6, §69]. If we set

$$\xi_1 = p_{12}\,,\quad \xi_2 = p_{31}\,,\quad \xi_3 = p_{23}\,,$$
$$\xi_4 = p_{03}\,,\quad \xi_5 = p_{02}\,,\quad \xi_6 = p_{01}\,,$$

we have that the lines of P^3 correspond to the points of the quadric Ω^4

$$\xi_1\xi_4 + \xi_2\xi_5 + \xi_3\xi_6 = 0$$

in P^5. Two lines of P^3 intersect exactly when their corresponding points on Ω^4 are conjugate, that is, the line joining these points lies entirely in Ω^4.

To a point x in P^3 we associate all lines of P^3 incident with x and hence a plane lying in Ω^4. To a plane u in P^3 we associated all lines of P^3 lying in u and hence a plane lying in Ω^4. These two families of planes doubly rule Ω^4. To a surface element or co-direction in P^3, that is, a point x and incident plane u, is then associated all lines of P^3 lying in u and incident with x. In Ω^4 this corresponds to the intersection of the planes corresponding to x and u and is a line. Hence, the 5-dimensional spaces of co-directions in P^3 and lines in Ω^4 correspond naturally.

Note that the co-direction in P^3 consisting of the point $x_0 = {}^t(1, 0, 0, 0)$ and the incident plane $u_0 : x_3 = 0$ in 3.2 corresponds to the line l_0 of Ω^4 joining the points ${}^t(0, 0, 0, 0, 1, 0)$ and ${}^t(0, 0, 0, 0, 0, 1)$ in 5.2. For, to the co-direction (x_0, u_0) is associated all lines of P^3 joining x_0 and a point $y = {}^t(y_0, y_1, y_2, 0)$ of u_0; such a line has Plücker coordinates

$$\begin{aligned} \xi_1 &= 0, & \xi_2 &= 0, & \xi_3 &= 0, \\ \xi_4 &= 0, & \xi_5 &= y_2, & \xi_6 &= y_1, \end{aligned}$$

and corresponds to a point of Ω^4 lying on l_0.

The projectivity g in $PSL(4; \mathbf{C})$ permutes the lines of P^3 by $x \wedge y \to gx \wedge gy$, a projectivity of P^5 which preserves Ω^4. In this way one obtains the isomorphism $A_3 \simeq D_3$:

$$PSL(4; \mathbf{C}) \simeq PSO(A; \mathbf{C}), \quad A = \begin{bmatrix} 0 & 1_3 \\ 1_3 & 0 \end{bmatrix},$$

[4, (25.8.4′)]. The spaces of co-directions in P^3 and lines in Ω^4 are homogeneous under $PSL(4; \mathbf{C})$ and $PSO(\mathrm{A}; \mathbf{C})$ respectively; hence the correspondence between these spaces is as homogeneous spaces. In fact, since (x_0, u_0) and l_0 correspond, their isotropy subgroups, as described in 3.2 and 5.2, correspond under the isomorphism.

From the isomorphism of the groups, we obtain the isomorphism of the Lie algebras $\mathfrak{sl}(4; \mathbf{C}) \simeq \mathfrak{o}(A; \mathbf{C})$, where X in $\mathfrak{sl}(4; \mathbf{C})$ is sent to the linear transformation $x \wedge y \to (Xx) \wedge y + x \wedge (Xy)$ in $\mathfrak{o}(A; \mathbf{C})$. With $X = (a_{ij})$, $i, j = 0, 1, 2, 3$, the matrix of this transformation with respect to the basis $e_i \wedge e_j$ is

$$\left[\begin{array}{ccc|ccc} a_{11}+a_{22} & -a_{23} & -a_{13} & 0 & a_{10} & -a_{20} \\ -a_{32} & a_{11}+a_{33} & -a_{12} & -a_{10} & 0 & a_{30} \\ -a_{31} & -a_{21} & a_{22}+a_{33} & a_{20} & -a_{30} & 0 \\ \hline 0 & -a_{01} & a_{02} & a_{00}+a_{33} & a_{32} & a_{31} \\ a_{01} & 0 & -a_{03} & a_{23} & a_{00}+a_{22} & a_{21} \\ -a_{02} & a_{03} & 0 & a_{13} & a_{12} & a_{00}+a_{11} \end{array}\right];$$

this describes the isomorphism explicitly. Under this isomorphism, the Lie algebras of the isotropy subgroups of (x_0, u_0) and l_0, as in 3.1 and 5.1, correspond. Moreover, the element

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{of } \mathfrak{sl}(4; \mathbf{C})$$

is sent into the element

$$\left[\begin{array}{ccc|c} & 0 & & 0 \\ \hline 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \end{array}\right] \quad \text{of } \mathfrak{o}(A; \mathbf{C}).$$

Since these are the root vectors for the maximal roots which determine the contact structures, as in 3.4 and 5.2, we conclude:

The 5-dimensional manifolds of co-directions in P^3 and lines in Ω^4 are isomorphic as algebraic homogeneous contact manifolds.

This isomorphism holds for the real contact manifolds also; cf. 3.5 and 5.2. The real connected centerless groups $PSL(4; \mathbf{R})$ and $PSO(A; \mathbf{R})$ are isomorphic; each consists of the elements fixed under complex conjugation of matrix entries.

5.4. The algebraic homogeneous contact manifolds of lines in the quadrics Ψ^{n+1} and Ω^{n+1}, 4.3 and 5.2, are isomorphic since they are both obtained from the simple complex Lie algebra of type B_l or D_l by the construction of 2.10. This isomorphism can be exhibited explicitly by means of a contact transformation which reduces to the line-sphere transformation, as described in Section 1, when $n = 3$.

Throughout, unprimed quantities refer to Ω^{n+1} and primed quantities to Ψ^{n+1}. Set $n + 3 = 2l + 1$ or $2l$ according as n is even or odd; $n \geqslant 2$.

Thus,

$$G = PSO(A;\ \mathbf{C}),\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & 1_l & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1_l \\ 1_l & 0 \end{bmatrix}$$

and

$$G' = PSO(A';\ \mathbf{C}),\ A' = \left[\begin{array}{c|ccc} 2\,.\,1_n & & 0 & \\ \hline & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & 0 & -1 & 0 \end{array}\right].$$

These are groups of projectivities preserving Ω^{n+1} and Ψ^{n+1}, respectively, in P^{n+2}.

In case n is odd, the transformation which we consider is

$$\begin{aligned}
\xi_1 &= \alpha_1 + \sqrt{-1}\,\alpha_2 & \xi_{l+1} &= \alpha_1 - \sqrt{-1}\,\alpha_2 \\
\xi_2 &= \alpha_3 + \sqrt{-1}\,\alpha_4 & \xi_{l+2} &= \alpha_3 - \sqrt{-1}\,\alpha_4 \\
&\ \vdots & &\ \vdots \\
\xi_{l-2} &= \alpha_{n-2} + \sqrt{-1}\,\alpha_{n-1} & \xi_{2l-2} &= \alpha_{n-2} - \sqrt{-1}\,\alpha_{n-1} \\
\xi_{l-1} &= \alpha_n + \lambda & \xi_{2l-1} &= \alpha_n - \lambda \\
\xi_l &= \mu & \xi_{2l} &= -\nu\,.
\end{aligned}$$

This is a projectivity of P^{n+2} which sends the quadric Ψ^{n+1}

$$\alpha_1^{\,2} + \dots + \alpha_n^{\,2} - \lambda^2 - \mu\nu = 0$$

into the quadric Ω^{n+1}

$$2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l} = 0\,.$$

In case n is even, the first equation of the transformation is $\xi_0 = \sqrt{2}\,\alpha_1$ and the remaining ones are like the above.

As before, we exhibit the details of the calculations for the case of n odd. For n even one need only carry along an additional initial row and column in the matrices; the conclusions are unchanged.

The matrix T of the transformation is

$$T = \begin{bmatrix} B & & 0 & & \\ & 1 & 1 & 0 & 0 \\ 0 & & & & \\ & 0 & 0 & 1 & 0 \\ \bar{B} & & 0 & & \\ & 1 & -1 & 0 & 0 \\ 0 & & & & \\ & 0 & 0 & 0 & -1 \end{bmatrix},$$

where B is the $(l-2)$ by $(2l-4)$ matrix

$$B = \begin{bmatrix} 1 \ \sqrt{-1} & & & \\ & 1 \ \sqrt{-1} & & 0 \\ & & \cdot & \\ 0 & & & \cdot \\ & & & \cdot \\ & & & & 1 \ \sqrt{-1} \end{bmatrix}$$

and $\bar{B}$ is its complex conjugate; T has inverse

$$T^{-1} = \frac{1}{2} \begin{bmatrix} {}^t\bar{B} & 0 & & {}^tB & 0 & \\ & 1 & 0 & & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ & 0 & 2 & & 0 & 0 \\ & 0 & 0 & & 0 & -2 \end{bmatrix}.$$

By direct calculation we ascertain the following:

(1) $A' = {}^tTAT$ and hence $G' = T^{-1}GT$. G and G' are conjugate, but do not coincide, in $PSL(n+3; \mathbf{C})$. As a consequence, $\mathfrak{g}' = T^{-1}\mathfrak{g}T$.

(2) $l_0' = T^{-1}l_0$; the line l_0 in Ω^{n+1} joining

$${}^t(0, \ldots, 0, 1, 0) \quad \text{and} \quad {}^t(0, \ldots, 0, 0, 1)$$

is sent to the line l_0' of Ψ^{n+1} joining

$${}^t(0, \ldots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad {}^t(0, \ldots, 0, 1 \mid -1, 0, 0).$$

Hence their isotropy subgroups, as in 5.2 and 4.3 are conjugate: $P' = T^{-1} P T$. As a consequence, $\mathfrak{p}' = T^{-1} \mathfrak{p} T$.

(3) The Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$ in 5.1 and 4.4 are conjugate: $\mathfrak{h}' = T^{-1} \mathfrak{h} T$. In fact, for

$$H = \operatorname{diag}(h_1, \ldots, h_l \mid -h_1, \ldots, -h_l)$$

in $\mathfrak{h}$, we have

$$T^{-1}HT = \operatorname{diag}\left[\begin{bmatrix} 0 & \sqrt{-1}h_1 \\ -\sqrt{-1}h_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \sqrt{-1}h_{l-2} \\ -\sqrt{-1}h_{l-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{bmatrix}\right]$$

in $\mathfrak{h}'$.

(4) The elements W and W' of the Lie algebras which give the contact structures on G/P and G'/P', as in 5.2 and 4.4, are conjugate: $W' = T^{-1} W T$. We conclude:

The $(2n-1)$-dimensional manifolds of lines in Ω^{n+1} and lines in Ψ^{n+1} are isomorphic as algebraic homogeneous contact manifolds. The isomorphism is a consequence of the projectivity T carrying Ψ^{n+1} into Ω^{n+1}. T sends lines of Ψ^{n+1} into lines of Ω^{n+1} and is a contact transformation.

5.5. $G_0 = PSO(A; \mathbf{R})$ is a real form of G; it consists of the elements of G fixed under the conjugation $g \to \bar{g}$ of G, complex conjugation of the matrix entries of g. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, as in 5.1, is stable and the maximal root $\rho = -\delta_{l-1} - \delta_l$ is real. With $P_0 = G_0 \cap P$, we obtain from 2.11 the real contact manifold

$$G_0/P_0 = \text{space of lines in } \Omega^{n+1} \text{ in real } P^{n+2},$$

a real form of G/P; cf. 5.2. The same remarks apply to the real form $G'_0 = PSO(A'; \mathbf{R})$ of G' for the conjugation $g' \to \bar{g}'$. With $P'_0 = G'_0 \cap P'$, we obtain the real contact manifold

$$\begin{aligned} G'_0/P'_0 &= \text{space of lines in } \Psi^{n+1} \text{ in real } P^{n+2} \\ &= \text{space of pencils of mutually tangent oriented spheres in real } E^n \\ &= \text{space of oriented co-directions in real } E^n, \end{aligned}$$

a real form of G/P; cf. 4.7.

Since $G' = T^{-1} G T$, we can exhibit G'_0/P'_0, as well as G_0/P_0, as a real form of the complex contact manifold G/P. $T G'_0 P^{-1}$ is the real form of $G = T G' T^{-1}$ consisting of the elements fixed under the conjugation obtained by transporting the conjugation $g' \to \bar{g}'$ of G' to G, namely

$$g \to T\,\overline{(T^{-1} g T)}\,T^{-1} = S^{-1} \bar{g} S$$

where $S = \bar{T} T^{-1}$. In the case of n odd,

$$S = \left[\begin{array}{cc|cc} 0 & 0 & 1_{l-2} & 0 \\ 0 & 1_2 & 0 & 0 \\ \hline 1_{l-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \end{array}\right];$$

in the case of n even, S has an additional initial row and column with a 1 in their common first entry and 0s elsewhere. ${}^t S A S = A$ and $S^2 = 1_{n+3}$, so the complex conjugation $\xi \to S^{-1}\bar{\xi}$ preserves the quadric Ω^{n+1}. A point or line of Ω^{n+1} is fixed under this conjugation exactly if it is the image under T of a real point or line of Ψ^{n+1}. The latter constitute the orbit on Ω^{n+1} of $T G'_0 T^{-1}$. The isotropy subgroup in $T G'_0 T^{-1}$ of the line l_0 of Ω^{n+1} is $T G'_0 T^{-1} \cap P = T P'_0 T^{-1}$. Furthermore, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in 5.1 is stable under the conjugation $X \to S^{-1} \bar{X} S$ of $\mathfrak{g}$; in fact, for

$$H = \text{diag}\,(h_1, \ldots, h_l \mid -h_1, \ldots, -h_l)$$

in $\mathfrak{h}$, we have

$$S^{-1} \bar{H} S = \text{diag}\,(-\bar{h}_1, \ldots, -\bar{h}_{l-2}, \bar{h}_{l-1}, \bar{h}_l \mid \bar{h}_1, \ldots, \bar{h}_{l-2}, -\bar{h}_{l-1}, -\bar{h}_l),$$

in case of n odd; the maximal root $\rho = -\delta_{l-1} - \delta_l$ is real, $\overline{\rho(S^{-1} \overline{H} S)} = \rho(H)$. Hence, the contact structure on $T G'_0 T^{-1}/T P'_0 T^{-1}$ is that obtained from G/P by 2.11. We conclude:

G_0/P_0 and $T G'_0 T^{-1}/T P'_0 T^{-1}$, the latter isomorphic to G'_0/P'_0, are two reals forms of the complex contact manifold G/P.

5.6. We observed in 5.3 that the space of co-directions in complex projective space P^3, by means of Plücker's line geometry, is isomorphic to the space of lines in the quadric Ω^4 in complex P^5, and that this isomorphism makes real line geometry correspond to a real form of Ω^4. We found in 5.4 and 5.5 that the space of oriented co-directions in complex Euclidean space E^3 of Lie's higher sphere geometry, which is the space of lines in the quadric Ψ^4 in complex P^5, is isomorphic to the space of lines in the quadric Ω^4 also, and that this isomorphism makes real sphere geometry correspond to a second real form of Ω^4. That is, real line geometry and real sphere geometry are two distinct real forms of complex line geometry. The line-sphere transformation establishes the isomorphism of the spaces of lines in Ψ^4 and lines in Ω^4. The former places real sphere geometry in the foreground, the latter, real line geometry.

5.7. The isomorphism of 5.3 may be used to describe sphere geometry in terms of co-directions in complex P^3. Real sphere geometry then leads to the real form $PSU(2,2)$ of $PSL(4; \mathbf{C})$.

REFERENCES

[1] Boothby, W. M. Homogeneous complex contact manifolds. *Proc. Sympos. Pure Math., Vol. III*, pp. 144-154, Amer. Math. Soc., Providence, R. I., 1961.

[2] —— A note on homogeneous complex contact manifolds. *Proc. Amer. Math. Soc. 13* (1961), pp. 276-280.

[3] Borel, A. *Linear Algebraic Groups.* W. A. Benjamin, Inc., New York, 1969.

[4] Freudenthal, H. and H. de Vries. *Linear Lie Groups.* Academic Press, New York, 1969.

[5] Klein, F. *Lectures on Mathematics, Lectures II and III: Sophus Lie.* The Evanston Colloquium, MacMillan and Co., 1893. Republished by Amer. Math. Soc., New York, 1911.

[6] —— *Vorlesungen über höhere Geometrie.* Springer-Verlag, Berlin 1926. 3. Aufl. reprinted by Chelsea, New York, 1957.