§3. Classification of acyclic map from a given space

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So $\pi_1(h)$ is injective and, by (1.3) and (1.5), h is a homotopy equivalence.

(2.6) THEOREM. *Let $f: X \to Y$ be an acyclic map between CW-spaces and let $h_1, h_2 : Y \to Z$ be two maps. If $h_1 f \simeq h_2 f$, then it follows that $h_1 \simeq h_2$.*

Proof. By (2.5) we have cofibre sequence

$$F \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \Delta F$$

where ΔF is the reduced suspension of the acyclic space F. Since ΔF is simply connected and $\tilde{H}_*(\Delta F) = 0$, it is contractible, and the group $[\Delta F, Z]$ in the Puppe sequence is zero.

In general, the group $[\Delta F, Z]$ acts transitively on the fibres of the function $[Y, Z] \to [X, Z]$, so that in this case, $[Y, Z] \to [X, Z]$ is injective. This proves the theorem.

§ 3. CLASSIFICATION OF ACYCLIC MAP FROM A GIVEN SPACE

Let X be a path connected space. To each acyclic map $f: X \to Y$, we assign the kernel of $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ which is a perfect normal subgroup of $\pi_1(X)$ by (1.3). The object of this section is to show that this map from isomorphism classes of acyclic maps defined on X to perfect normal subgroups of $\pi_1(X)$ is a bijection.

(3.1) PROPOSITION. *Let $f: X \to Y$ and $f' : X \to Y'$ be two maps between CW-spaces such that f is acyclic. There exists a map $h : Y \to Y'$ with $hf \simeq f'$ if and only if* $\ker \pi_1(f) \subset \ker \pi_1(f')$, *and such an h is unique up to homotopy. In addition, if f' is acyclic, then h is acyclic, and h is a homotopy equivalence if and only if* $\ker \pi_1(f) = \ker \pi_1(f')$.

Proof. If h exists, then $\pi_1(f') = \pi_1(h) \circ \pi_1(f)$ and we have $\ker \pi_1(f) \subset \ker \pi_1(f')$. Conversely, we can suppose f is a cofibration and form the cocartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{g} & Y' \cup_X Y \end{array}$$

where g is an acyclic map by (2.3). Now calculate

$$\pi_1(g) : \pi_1(Y') \to \pi_1(Y' \cup_X Y) = \pi_1(Y')_{*\pi_1(X)} \pi_1(Y)$$

by the vanKampen theorem. Since $\ker(\pi_1(f)) \subset \ker(\pi_1(f'))$, it follows that $\pi_1(g)$ is an isomorphism, and by (1.5) the map g is a homotopy equivalence. Let $g^* : Y' \cup_X Y \to Y'$ be a homotopy inverse of g. Then $h = g^* g' : Y \to Y'$ is the desired map with $hf = f'$. The map h is unique by (2.6).

The map h is acyclic by (2.1). Since $\pi_1(h)$ is an isomorphism if and only if $\ker \pi_1(f) = \ker \pi_1(f')$, the last statement follows from (1.5), and this proves the proposition.

(3.2) COROLLARY. *Let A be an acyclic CW-space. A map $f : A \to Z$ is null homotopic if and only if $\pi_1(f)$ is zero.*

Proof. We apply (3.1) to the acyclic map $A \to *$, and when $\pi_1(f)$ is zero, f factors $A \to * \to Z$ up to homotopy.

(3.3) PROPOSITION. *Let X be a path connected space, and let N be a perfect normal subgroup of $\pi_1(X)$. Then there exists an acyclic map $f : X \to Y$ with* $\ker \pi_1(f) = N$. *If X has the homotopy type of a CW-complex, then so does Y.*

Proof. First, we do the case where $N = \pi_1(X)$ is perfect. Let T_1 be a wedge of circles indexed by generators of N and $u : T_1 \to X$ a map such that $\pi_1(u)$ is surjective. We form the cofibre $v : X \to X^*$ of u, i.e. attach a 2-cell for each circle. By the van Kampen theorem it follows that $\pi_1(X^*) = 0$ and the homology exact sequence of the cofibration takes the form

$$0 \to H_q(X) \to H_q(X^*) \to 0 \quad \text{for} \quad q \geqq 3$$

and

$$0 \to H_2(X) \to H_2(X^*) \xrightarrow{\partial} H_1(T) \to H_1(X) = 0\,.$$

Since $H_1(T_1)$ is free abelian, it lifts back into $H_2(X^*)$, and since $\pi_2(X^*) \to H_2(X^*)$ is an isomorphism by the Hurewicz theorem, there is a wedge T_2 of two spheres and a map $w : T_2 \to X^*$ such that $\partial H_2(w) : H_2(T_2) \to H_1(T_1)$ is an isomorphism. Let $X^* \to Y$ denote the cofibre of $w : T_2 \to X^*$, and let $f : X \to Y$ denote the composite $X \to X^* \to Y$. The cofibration homology exact sequence takes the form

$$0 \to H_q(X^*) \to H_q(Y) \to 0 \quad \text{for} \quad q \geqq 4,\ q = 1$$

and

$$
\begin{array}{ccccccccccc}
 & & & & & & H_1(T_1) & & & & \\
 & & & & & \nearrow \wr & \uparrow & & & & \\
0 \to & H_3(X^*) & \to & H_3(Y) & \to & H_2(T_2) & \to H_2(X^*) & \to & H_2(Y) & \to 0 & \\
 & & & & & & \uparrow & \nearrow H_2(f) & & & \\
 & & & & & & H_2(X) & & & &
\end{array}
$$

From this, a quick examination of the homology sequence reveals that $H_*(f) : H_*(X) \to H_*(Y)$ is an isomorphism. Since Y is simply connected, every local system on Y is trivial, and $H_*(f)$ is an isomorphism for all coefficients. By (1.2) (c) the map f is an acyclic map with the desired properties.

For a general perfect normal subgroup $N \subset \pi_1(X)$, let $g : \tilde{X}_N \to X$ be the covering corresponding to N, that is, $\operatorname{im}(\pi_1(g)) = N$, and let $f_0 : \tilde{X}_N \to Y_0$ be the acyclic map with $\ker(\pi_1(f_0)) = N = \pi_1(X_0)$ constructed in the previous paragraph. Change it up to homotopy into a cofibration, and form the following cocartesian diagram.

$$
\begin{array}{ccc}
X_N & \xrightarrow{f_0} & Y_0 \\
g \downarrow & & \downarrow \\
X & \xrightarrow{f} & X \cup_{\tilde{X}_N} Y_0
\end{array}
$$

By (2.3) the map f is acyclic. In order to calculate, we determine, using the van Kampen theorem, the group $\pi_1(X \cup_{\tilde{X}_N} Y_0) \cong \pi_1(X) * \pi_1(\tilde{X}_N)$ $\pi_1(Y_0)$. Since $\pi_1(Y_0) = 1$, it follows that $\pi_1(X \cup_{\tilde{X}_N} Y_0) \cong \pi_1(X)/\pi_1(\tilde{X}_N) = \pi_1(X)/N$. The morphism $\pi_1(f)$ is thus an epimorphism with kernel N. This proves the proposition.

(3.4) DEFINITION. *Two acyclic maps* $f: X \to Y$ *and* $f' : X \to Y'$ *defined on* X *are equivalent provided there exists a homotopy equivalence* $h : Y \to Y'$ *with* $hf \simeq f'$.

Putting together propositions (3.1) and (3.3), we obtain the classification theorem.

(3.5) THEOREM. *Let* X *be a path connected space with the homotopy type of a CW-complex. The function which assigns to an acyclic map*

$f: X \to Y$ the subgroup $\ker(\pi_1(f))$ *of* $\pi_1(X)$ *is a bijection from the set of equivalence classes of acyclic maps on* X *to the set of normal perfect subgroups of* $\pi_1(X)$.

Proof. The function is injective by (3.1) and surjective by (3.3).

In view of this theorem we see that the theory of acyclic maps is similar to the theory of covering spaces, in that, they are classified by certain subgroups of the fundamental group. By way of comparison, for covering maps $f: Y \to X$ over X, the group $\operatorname{im} \pi_1(f)$ is given, and $\pi_q(f)$ is an isomorphism for $q \geqq 2$. The homology of Y is related to that of X by a spectral sequence. For acyclic maps $f: X \to Y$ from X, the group $\ker \pi_1(f)$ classifies the objects. It is perfect and normal, and $f_*: H_*(X, f^{-1}L) \to H_*(Y, L)$ is an isomorphism for any local system L on Y. The higher homotopy groups of X and Y are not easily related in general (but see § 5).

(3.6) *Notations.* Let $\mathscr{P}$ be the category whose objects are pairs (X, N) where X is a pointed CW-space and N is a perfect normal subgroup of $\pi_1(X)$ and whose morphisms $f: (X, N) \to (X', N')$ are homotopy classes of maps $f: X \to X'$ with $\pi_1(f)(N) \subset N'$. Let (CW) be the category of pointed CW-spaces and homotopy classes of maps. We have two natural functors $\alpha: (CW) \to \mathscr{P}$ and $\beta: \mathscr{P} \to (CW)$ with $\beta\alpha$ the identity where $\beta(X, N) = X$ and $\alpha(X) = (X, N_0)$ for N_0 the maximal normal perfect subgroup of $\pi_1(X)$.

(3.7) THEOREM. *For* (X, N) *in* $\mathscr{P}$ *choose* $f: X \to X_N^+$ *an acyclic map with* $\ker(\pi_1(f)) = N$. *Then there is a functor* $\sigma: \mathscr{P} \to (CW)$ *and a morphism of functors* $f: \beta \to \sigma$ *such that* $\sigma(X, N) = X_N^+$ *and* $f(X, N) = f$.

Proof. This immediate from the universal property (3.1).

(3.8) *Remark.* The space X_N^+ is unique up to homotopy equivalence. The acyclic map $X \to X_N^+$ we had to choose is defined up to the composition with a homotopy equivalence of X_N^+. However, we shall give in Section 4 a stronger functorial way to construct acyclic maps without any choice, for instance the functorial plus construction $f: X \to X^+$ where $X^+ = \sigma\alpha(X)$