Application 4 (The spectral theorem for normal operators). Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the Banach algebra of all bounded linear operators on $\mathscr{H}$. Consider a subalgebra $A \subset \mathscr{L}(\mathscr{H})$ with the following properties:

(i) A is commutative;

(ii) A is closed;

(iii) If $T \in A$ then $T^* \in A$;

(iv) The identity operator belongs to A.

Let Δ denote the maximal ideal space of A. Since each $T \in A$ is normal it follows that $\|T\| = \|\hat{T}\|_\infty$ for every $T \in A$.

For each pair of vectors $\xi, \eta \in \mathscr{H}$ define a mapping $L_{\xi,\eta} : A \to \mathbf{C}$ by

$$L_{\xi,\eta}(T) = (T\xi, \eta)$$

then we have

$$|L_{\xi,\eta}(T)| \leqslant \|T\| \cdot \|\xi\| \; \|\eta\| = \|\xi\| \; \|\eta\| \cdot \|\hat{T}\|_\infty$$

Therefore by Theorem 1 there exists a bounded complex Radon measure $\mu_{\xi,\eta}$ on Δ such that $\|\mu_{\xi,\eta}\| \leqslant \|\xi\| \cdot \|\mu\|$ and

$$L_{\xi,\eta}(T) = \int_\Delta \hat{T} \, d\mu_{\xi,\eta} \quad \text{for every} \quad T \in A .$$

An application of the Gelfand-Neumark theorem establishes the uniqueness of the measure. The usual construction of a unique resolution of the identity on the Borel sets of Δ can be made based on this formula. A specialization of this formula to a single normal operator leads to the classical spectral theorem. We shall not give the details here since many excellent accounts exist (c.f. Berberian [1], [2], Segal-Kunze [18]). An especially lucid presentation is given in Rudin [16].

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