8. Infinitely Fine Partitions of an interval

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Let α be an infinite natural number. By the above theorem we have $a_\alpha \approx a$ and $b_\alpha \approx b$. From this we see easily that a_α and b_α are finite. Now using the rules given in Section 2 for manipulating the $\approx$ symbol,

$$a_\alpha + b_\alpha \approx a + b \text{ and } a_\alpha\, b_\alpha \approx a\, b.$$

Thus by the above theorem, the desired results are established.

Example 7.2. Suppose we wanted to calculate

$$\lim_{n \to \infty} (n^2 - n) = ?$$

We can proceed directly— let α be an arbitrary infinite natural number, then

$$\begin{aligned} \alpha^2 - \alpha = \alpha(\alpha - 1) &= \text{(infinite) (infinite)} \\ &= \text{infinite} \end{aligned}$$

thus

$$\lim_{n \to \infty} (n^2 - n) = \infty.$$

8. Infinitely Fine Partitions of an Interval

Consider the familiar process of partitioning an interval $[a, b]$ into n subintervals of equal length by means of the partition points

$$a = a_0 < a_1 < \cdots < a_n = b.$$

If we let a_i^j denote the i^{th} partition point when the interval is divided into j subintervals of equal length, it is easily seen that

$$a_i^j = a + \left(\frac{b-a}{j}\right) i\,.$$

Now the right side of this expression is a function from $I \times I$ into R, where $I \subseteq R$ is the set of integers. By the Main Theorem this function extends to a function from $I^* \times I^*$ into R^*. We continue to use a_i^j for the image under this extended function. If we let α be a fixed infinite natural number, then for $0 \leqq i \leqq \alpha$, a_i^α must lie in the interval $[a, b]^*$. Note that the i^{th} subinterval $[a_i^\alpha, a_{i+1}^\alpha]$ has the infinitesimal $\dfrac{b-a}{\alpha}$ as its length. Two such intervals can intersect only if they have an end point in common, and the intersection is that end point. Each partition point a_i^α (other than a, b) has an immediately preceding partition point a_{i-1}^{α} on its left and an immediately succeeding partition point a_{i+1}^{α} on its right. One can show that each point of $[a, b]^*$ appears in some subinterval $[a_i^{\alpha}, a_{i+1}^{\alpha}]$ as follows. Formulate as an admissible statement (true in R) the assertion:

"For every $j \in N$, every point of $[a, b]$ is in the subinterval $[a_i^j, a_{i+1}^j]$ for some $i \in I$ where $0 \leqq i < j$."

Putting in appropriate stars *, it becomes true in R^*. Particularizing it to the case where $j = \alpha$ we get:

"Every point of $[a, b]^*$ is in the subinterval $[a_i^{\alpha}, a_{i+1}^{\alpha}]$ for some $i \in I^*$ where $0 \leqq i < \alpha$."

Using the above we can now show that the partition described there has uncountably many points from which it also follows that $\{ \beta \in N^* \mid \beta \leqq \alpha \}$ and N^* are uncountable. We do this by showing a mapping from $\{ a_0^{\alpha}, a_2^{\alpha}, \ldots, a_{\alpha}^{\alpha} \}$ onto the standard interval $[a, b]$ which is known to be uncountable. Each partition point a_i^{α} being finite is infinitely close (Theorem 2.1) to a uniquely determined real. Let the image of a_i^{α} be that real. Clearly the image is in $[a, b]$. Moreover the mapping is onto because we saw that each real c in $[a, b]$ is a member of $[a_i^{\alpha}, a_{i+1}^{\alpha}]$ some $0 \leqq i < \alpha$, and since $a_i^{\alpha} \approx a_{i+1}^{\alpha}$, we must also have $c \approx a_i^{\alpha}$.

Consider the following novel proof of a famous theorem.

THEOREM 8.1. If the standard function f is continuous on the standard interval $[a, b]$ and is negative at a and positive at b, then at some standard point c in the interval, $f(c) = 0$.

PROOF. Let α be an infinite natural number and form the infinitely fine partition $\{ a_0^{\alpha}, a_2^{\alpha}, \ldots, a_{\alpha}^{\alpha} \}$ described earlier in this section. Now the following assertion can be formulated as an admissible statement true in R:

"For each $j \in N$ there exists a least $i \in N$ such that $0 < i \leqq j$ and $f(a_i^j) \geqq 0$."

Putting in stars this becomes true in R^*. Now particularizing it to the case $j = \alpha$ we get (leaving off some stars for brevity):

"Exists least $i \in N^*$ such that $0 < i \leqq \alpha$ and $f(a_i^{\alpha}) \geqq 0$."

For this i then we must have $f(a_{i-1}^{\alpha}) < 0$. Now a_i^{α} is finite and must be infinitely close to a standard number c in the interval. Since f is a standard function, $f(c)$ is standard. Now from $a_{i-1}^{\alpha} \approx a_i^{\alpha}$ we get

$$c \approx a_i^{\alpha} \text{ and } c \approx a_{i-1}^{\alpha}.$$

Then by continuity we see that

$$f(c) \approx f(a_i^\alpha) \text{ and } f(c) \approx f(a_{i-1}^\alpha).$$

Taking this together with the fact (seen already) that

$$f(a_i^\alpha) \geqq 0 \text{ and } f(a_i^\alpha) < 0$$

we have (in summary) that $f(c)$ is a standard number infinitely close to a negative number and a non-negative number. Thus $f(c) = 0$.

(Q.E.D.)

9. Derivatives

Let $f(x)$ be a standard function defined on a standard open interval (a, b) and having the point x_0 as an interior point. Using the non-standard characterization of limit, the condition that $f(x)$ be differentiable at x_0 is that there exist a standard number L such that

$$\frac{f(x_0 + dx) - f(x_0)}{dx} \approx L$$

for all non-zero infinitesimals dx. L, of course, will be the derivative. If $f(x)$ is differentiable, then writing $dy = f(x_0 + dx) - f(x_0)$ we have (using the notation for "standard part" introduced in Section 2) $°(\frac{dy}{dx}) = f'(x_0)$. This says that the quotient of the infinitesimal increments need not in general be the derivative, but it must be infinitely close to it.

Example 9.1. Suppose we wish to calculate the derivative of $f(x) = x^2$. Let dx be an arbitrary non-zero infinitesimal, then

$$\frac{dy}{dx} = \frac{(x + dx)^2 - x^2}{dx}$$

After squaring and cancelling we get, $\frac{dy}{dx} = 2x + dx \approx 2x$ therefore

$$°(\frac{dy}{dx}) = 2x.$$

That is, the function x^2 is differentiable with derivative $2x$.

Example 9.2. Let's see how to prove the Chain Rule! Suppose $f(x)$ and $g(x)$ are differentiable at the appropriate places and we wish to show