

# 5. Infinite Natural Numbers

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For any finite set  $T \subseteq R$  we can show  $T = T^*$ . Suppose  $T = \{a_1, \dots, a_n\}$  then

$$(\forall x) (x \in T \leftrightarrow [x = a_1 \vee x = a_2 \vee \dots \vee x = a_n])$$

is true in  $R$ , thus

$$(\forall x) (x \in T^* \leftrightarrow [x = a_1 \vee x = a_2 \vee \dots \vee x = a_n])$$

is true in  $R^*$ ; that is,  $T^* = \{a_1, \dots, a_n\}$ .

Although the Main Theorem makes no mention of functions  $f$  whose domain is a proper subset  $D \subset R$ . We can define a function  $f^* : D^* \rightarrow R^*$  in a natural way. Arbitrarily extend  $f$  to a function  $g$  which is defined on all of  $R$ ; then let  $f^*$  be the restriction of  $g^*$  to  $D^*$ . This definition is easily seen to be independent of the way  $f$  is extended.

## 5. INFINITE NATURAL NUMBERS

We have seen in the last section that each particular  $S \subseteq R$  has associated with it a certain extension  $S^* \subseteq R^*$ . We now consider the case when we take  $S$  to be  $N$ , the set of natural numbers. One can see that  $N^*$  actually has some non-standard members as follows. The statement “ $N$  is unbounded” is true in  $R$  and can be formulated as the admissible statement

$$(\forall x) (\exists y) (y \in N \wedge y > x);$$

therefore

$$(\forall x) (\exists y) (y \in N^* \wedge y > x)$$

is true in  $R^*$ . It asserts that  $N^*$  is an unbounded subset of  $R^*$ . If we let  $\alpha$  be an infinite member of  $R^*$ , then  $N^*$  must have an even larger member which, of course, is also infinite and non-standard.

We can show that all the non-standard members of  $N^*$  are infinite in the following way. Formulate as admissible statements each of the infinitely many assertions:

“All natural numbers are greater than 0.”

“No natural numbers lie between 0 and 1.”

“No natural numbers lie between 1 and 2.”

etc.

Each of those statements then must be true in  $R^*$  when we read  $N^*$  instead of  $N$ , so each member of  $N^* - N$  must be greater than all the real numbers.

In view of the above we call the non-standard members of  $N^*$  *infinite natural numbers*.

Now it is easy to show that each infinite natural number has an immediate successor in  $N^*$  (because of the corresponding result for  $N$ ), and each infinite natural number has an infinite immediate predecessor in  $N^*$ .  $N^*$  isn't well ordered because if  $\alpha$  is an infinite natural number, the chain

$$\alpha > \alpha - 1 > \alpha - 2 > \dots$$

has no least member. Here again one might be tempted to use the Main Theorem to infer that  $N^*$  is well ordered because  $N$  is; however, the statement that  $N$  is well ordered is not admissible by virtue of its having a variable ranging over subsets. It reads:

“Every non-empty subset of  $N \dots$ ”

Concepts such as even number, odd number, and prime number are all meaningful for infinite natural numbers; indeed, if  $E \subseteq N$  is the set of even numbers, then  $E^*$  is the set of even numbers of  $N^*$ .

It will be shown later that  $N^*$  is uncountably infinite.

## 6. LIMITS, CONTINUITY, BOUNDEDNESS, AND COMPACTNESS

Now we show that  $R^*$  provides the appropriate machinery for formulating concepts from the Calculus in an intuitive and direct way. Consider, for example, the limit concept. The  $\varepsilon - \delta$  definition of  $\lim_{x \rightarrow c} f(x) = L$  seems to be a roundabout way of saying that for  $x$  infinitely close to but not equal to  $c$ ,  $f(x)$  will be infinitely close to  $L$ . Now it makes sense to say it just that way provided we are talking about  $f^*(x)$ . It not only makes sense, but as the next theorem shows, saying it that way actually gives a correct characterization of  $\lim_{x \rightarrow c} f(x) = L$ .

**THEOREM 6.1.** Let  $f$  be a standard function defined on a standard open interval  $(a, b)$  having  $c$  as an interior point. Suppose further that  $L$  is standard, then

(a)  $\lim_{x \rightarrow c} f(x) = L$  if and only if  $c \neq x \approx c$  implies  $f^*(x) \approx L$ .