

3. The Main Theorem

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3. THE MAIN THEOREM

All that has been said here so far was known long before the advent of Non-Standard Analysis. Now we come to the heart of the matter—the key theorem. It was first obtained by Robinson as a corollary to the so-called Compactness Theorem of mathematical logic. Later proofs were given by means of the ultraproduct construction which also has its roots in mathematical logic. We shall content ourselves with a mere statement of the result. R as usual denotes the real number system and N the natural number system.

THEOREM 3.1 (MAIN THEOREM). There is a set R^* for which all of the following hold:

1. R is a proper subset of R^* .
2. To each n -place function $f(x_1, \dots, x_n)$ from R^n to R ($n \geq 1$), there corresponds a certain function $f^*(x_1, \dots, x_n)$ from $(R^*)^n$ to R^* which agrees with $f(x_1, \dots, x_n)$ on R^n .
3. To each n -place relation $A(x_1, \dots, x_n)$ on R ($n \geq 1$), there corresponds a certain relation $A^*(x_1, \dots, x_n)$ on R^* which agrees with $A(x_1, \dots, x_n)$ on R . The relation corresponding to the equality relation on R is the equality relation on R^* .
4. Every statement $\mathcal{S}$ formulated in terms of
 - i) particular (fixed) real numbers
 - ii) particular (fixed) real functions
 - iii) particular (fixed) real relations
 - iv) variables ranging over R
 - v) logical operations and quantifiers
 is true about R if and only if the statement $\mathcal{S}^*$ obtained from it by
 - a) replacing each $f(x_1, \dots, x_n)$ by $f^*(x_1, \dots, x_n)$
 - b) replacing each $A(x_1, \dots, x_n)$ by $A^*(x_1, \dots, x_n)$
 - c) letting the variables range over R^*
 is true about R^* .

It turns out that there are many such R^* . From here on out it will be assumed that we are fixing on one of them.

The theorem is quite a mouthful and it must be admitted that our formulation of it suffers from a little imprecision owing to the fact that we

never said what a statement is ¹⁾. A few examples, however, should nail the idea down. Let us add for emphasis that we are only allowing statements of finite length.

Example 3.1. Consider the statement

$$(\forall x) (0 + x = x)$$

which is true when the variables range over R ; it asserts that the particular real number 0 is a left identity for the $+$ operation. By the Main Theorem, the statement

$$(\forall x) (0 +^* x = x)$$

must be true when the variables range over R^* ; thus 0 is also a left identity for the $+^*$ operation on R^* .

Example 3.2. Let f be a particular function from R to R which is an “onto” function. Then the statement

$$(\forall y) (\exists x) (f(x) = y)$$

is true when the variables range over R . Therefore by the Main Theorem the statement

$$(\forall y) (\exists x) (f^*(x) = y)$$

is true when the variables range over R^* ; that is, the function f^* is onto R^* .

Henceforth instead of saying “true when the variables range over R ”, we shall simply say “true in R ”.

In subsequent discussions members of R will be called *standard* numbers, while members of $R^* - R$ will be called *non-standard* numbers. Likewise functions from R^n to R ($n \geq 1$), relations on R , and subsets of R will be called *standard* functions, relations and subsets. Some writers refer to members of R^* as real numbers, but we shall reserve the term for members of R . Thus standard number and real number have the same meaning here.

Statements which can be formulated in the manner prescribed in the hypothesis of the Main Theorem are called *admissible* statements. You should convince yourself, by writing them out if necessary, that all the axioms of an ordered field are admissible; moreover, they are true about R (because

¹⁾ Using the terminology of formal logic the class of statements in question can be defined as the class of closed well-formed formulae of a generalized first-order language having distinct individual, function, and relation constants corresponding to each real, real function and real relation.

R is an ordered field). Now by the Main Theorem they are all true about R^* if we put the stars on the symbols $+$, $\times$, $<$. But this is just a way of saying that R^* is an ordered field with respect to $+^*$, $\times^*$, $<^*$. Moreover since the theorem provided that these agree with $+$, $\times$, $<$ respectively on R , we can say that R^* is an ordered field which has R as a proper subordered field. Now recalling results from our review on ordered fields we have that R^* is non-Archimedean and is not complete.

Now at this point you might be getting a bit suspicious. You might ask: "Why not show the completeness of R^* (and thus get a paradox) by taking the assertion that R is complete, and then use the Main Theorem to conclude that R^* is complete?" The catch is that the Completeness Axiom has a logical structure fundamentally different from the ordered field axioms. It's not an admissible statement! Its form is

$$(\forall S) (S \text{ bounded} \rightarrow \dots\dots\dots)$$

that is, it has a variable ranging over the family of *subsets* of R . Recall, the variables in an admissible statement must range over R .

With respect to the Archimedean property the catch is a little different. Using the symbols $N(y)$ to denote the particular one-place relation—"y is a natural number," we *can* assert that R is Archimedean by the admissible statement

$$(\forall x) (\exists y) (N(y) \wedge x < y);$$

thus

$$(\forall x) (\exists y) (N^*(y) \wedge x <^* y)$$

is true in R^* , but it doesn't necessarily say that R^* is Archimedean. The y which is asserted to exist, and for which $N^*(y)$ holds, might be in $R^* - R$; that is, it might be non-standard. To be sure, it does say that R^* has some sort of formal Archimedean-like property, but if in the definition of Archimedean one requires that y actually be a member of N (and we shall), then R^* isn't Archimedean.

In the sequel it may at times be too repetitious to write statements first without the stars * , and then with them. It will usually be clear from the context whether the stars are intended. Thus if we were to say that

$$(\forall x) (\forall y) (x < y \rightarrow f(x) < f(y))$$

is true in R^* , then you are to understand that we are really talking about

$<^*, f^*$ and the variables are to range over R^* . Sometimes we shall put on some of the stars for emphasis.

4. FIXED SUBSETS

Let S be a particular (fixed) subset of R . We can identify S with the one-place relation $S(x)$ which holds for a given x if and only if $x \in S$; that is,

$$S = \{ x \in R \mid S(x) \}.$$

We can now define a set $S^* \subseteq R^*$ by

$$S^* = \{ x \in R^* \mid S^*(x) \}.$$

Clearly $S \subseteq S^*$ because $S^*(x)$ agrees with $S(x)$ on R . We shall often write

$$x \in S \text{ instead of } S(x)$$

and

$$x \in S^* \text{ instead of } S^*(x).$$

The upshot of the above is that the Main Theorem also provides for an extension S^* for each $S \subseteq R$ and that we can allow as admissible statements those which involve the sentence fragment $x \in S$; in “lifting” statements from R to R^* we replace the fragment $x \in S$ by $x \in S^*$. Warning! The requirement that admissible statements be permitted only variables ranging over R hasn’t been altered. In a given statement the functions, relations, and subsets must remain fixed!

Example 4.1. Let $S = \{ x \in R \mid x < 6 \}$. Now

$$(\forall x) (x \in S \leftrightarrow x < 6) \text{ is true in } R$$

so

$$(\forall x) (x \in S^* \leftrightarrow x <^* 6) \text{ is true in } R^*.$$

Thus

$$S^* = \{ x \in R^* \mid x <^* 6 \}.$$

Furthermore S^* is a proper extension of S , because for any infinitesimal ε , the number $5 + \varepsilon$ is a member of S^* , but not being a standard number, it can’t be a member of S .