

# Concluding Remarks

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **15 (1969)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

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From (4.6) and (4.7) we obtain, appealing first to Theorem A and then to Lemma 3,

$$A(x) = o(x^{\alpha+1+\delta}), \quad A^k(x) = o(x^{k+\alpha+1+\delta}), \quad 0 \leq k < r. \quad (4.8)$$

Now Lemma 2 establishes the summability  $(R, l_n, k)$  of  $\Sigma a_n l_n^{-(\alpha+1+\delta)}$ , or, of  $\Sigma a_n l_n^{-s}$  for  $\sigma \geq \alpha+1+\delta$  with arbitrary  $\delta > 0$ . Hence  $\sigma_k \leq \alpha+1$  as required.

(B) We now choose  $\gamma$  such that  $(\alpha+1 \leq) \sigma_r < \gamma$  and note that  $\alpha+1+\delta$  can be replaced by  $\gamma$  in (4.7) and (4.8), so that, arguing as before, we establish the summability  $(R, l_n, k)$ ,  $0 \leq k < r$ , of  $\Sigma a_n l_n^{-\gamma}$  where  $\gamma > \sigma_r$  is arbitrary. Hence  $\sigma_k \leq \sigma_r$  while  $\sigma_r \leq \sigma_k$  universally, i.e.,  $\sigma_k = \sigma_r$  as we wished to prove.

DEDUCTION 3. *If, for the Dirichlet series  $\Sigma a_n l_n^{-s}$ ,  $\sigma_r > -\infty$  and  $\lim l_n/l_{n-1} > 1$ , then  $\sigma_k = \sigma_r$  for  $0 \leq k < r$ .*

*Proof.* The hypothesis  $\lim l_n/l_{n-1} > 1$  makes

$$a_{n+1} + a_{n+2} + \dots + a_m = 0 \quad \text{for} \quad l_n < l_m < l_n + \varepsilon l_n$$

if  $\varepsilon$  is sufficiently small and  $n > n_0(\varepsilon)$ . Hence, for any  $\rho$ , in particular, for  $\rho \leq \sigma_r$ ,

$$\lim_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^\rho} = o(1), \quad \varepsilon \rightarrow 0.$$

The desired conclusion now follows from Theorem I (B) with alternative (2.4) (b).

In the above proof we have supposed that  $\sigma_r < \infty$ , the case  $\sigma_r = \infty$  being trivial.

### CONCLUDING REMARKS

A few remarks are offered in conclusion, supplementing some made in the beginning. Though Theorem A in one form is Karamata's (as already said), a particularization of it ([12], Corollary VI with Tauberian  $O$ -condition) is a much older theorem of Ananda-Rau's ([1], Theorem 16; [2], Theorem 4). Ananda Rau left open one case of his theorem which Bosanquet ([4], Theorems 2, 3), Minakshisundaram and Rajagopal ([10], Theorem 1 and Corollaries 1.1, 1.3; [11], Theorem A and Corollaries  $A_1, A_2$ ) have independently settled, even for some extensions of Ananda Rau's theorem. The theorem mentioned at the outset as being due to Chandrasekharan and Minakshisundaram ([6], p. 21, Theorem 1.82) is, in fact, a further extension of one of the extensions of Ananda Rau's theorem given by

Bosanquet ([4], Theorem 3). In the present context, it is rather less effective than the completely independent two-fold result of Karamata's in the same direction ([9], Théorèmes 1a), 3f)), reformulated as Theorem A. That is to say, precisely, Theorem A gives rise to a basic converse theorem on abscissae of summability of general Dirichlet series (Theorem I of this paper) which is more natural and suggestive as well as more comprehensive than the like basic theorem resulting from the line of development followed by Chandrasekharan and Minakshisundaram ([6], p. 86, Theorem 3.71).<sup>1)</sup>

I am indebted to Prof. Bosanquet for some very useful remarks on the original version of this paper which have led to the preparation of the present version.

#### REFERENCES

- [1] ANANDA-RAU, K., *On some properties of Dirichlet's series*. Smith's Prize Essay, Cambridge 1918.
- [2] ——— On the convergence and summability of Dirichlet's series. *Proc. London Math. Soc.*, (2), 34 (1932), 414-440.
- [3] BOSANQUET, L. S., On the summability of Fourier series. *Proc. London Math. Soc.*, (2), 31 (1931), 144-164.
- [4] ——— Note on convexity theorems. *J. London Math. Soc.*, 18 (1943), 239-248.
- [5] ——— Note on the converse of Abel's theorem. *J. London Math. Soc.*, 19 (1944), 161-168.
- [6] CHANDRASEKHARAN, K. and S. MINAKSHISUNDARAM, *Typical Means* (Tata Institute of Fundamental Research Monographs in Mathematics and Physics, No. 1), Bombay 1952.
- [7] GANAPATHY IYER, V., Tauberian and summability theorems on Dirichlet's series. *Ann. of Math.*, 36 (1935), 100-116.
- [8] KARAMATA, J., On an inversion of Cesàro's method of summing divergent series (Serbian). *Glas. Srpske Akad. Nauk*, 191 (1948), 1-37.
- [9] ——— Quelques théorèmes inverses relatifs aux procédés de sommabilité de Cesàro et Riesz. *Acad. Serbe Sci. Publ. Inst. Math.*, 3 (1950), 53-71.
- [10] MINAKSHISUNDARAM, S. and C. T. RAJAGOPAL, An extension of a Tauberian theorem of L. J. Mordell. *Proc. London Math. Soc.*, (2), 50 (1945), 242-255.
- [11] ——— and C. T. RAJAGOPAL, On a Tauberian theorem of K. Ananda Rau. *Quart. J. Math. Oxford Ser.*, 17 (1946), 153-161.
- [12] RAJAGOPAL, C. T., On Tauberian theorems for the Riemann-Liouville integral. *Acad. Serbe Sci. Publ. Inst. Math.*, 6 (1954), 27-46.

(Reçu le 15 Juillet 1968)

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<sup>1)</sup> Indeed the Chandrasekharan-Minakshisundaram theorem just referred to is deducible from Theorem I, its case  $\sigma_r < \alpha + \mu$  [or, case  $\sigma_r \geq \alpha + \mu$ ] from part (A) [or, part (B)] of Theorem I with hypothesis (2.2) (b) and  $x^\rho = x^\alpha \{ \theta(x) \}^\mu$ ,  $\theta(x) = x^{(r-\alpha+\gamma)/(r+\mu)}$ ,  $\sigma_r < \gamma < \alpha + \mu$  [or, hypothesis (2.4) (b) and  $x^\rho = x^{\alpha+\mu}$ ].