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THE ASYMPTOTIC BEHAVIOR OF THE n^{th} ORDER DIFFERENCE

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To the memory of J. Karamata

The fundamental result on the regularly varying functions, proved originally by Karamata [1], [2] for continuous functions, and later by Korevaar, van Aardenne-Ehrenfest and de Bruijn [3] for measurable functions, can be stated as follows: If f is a positive measurable function and if $\frac{f(tx)}{f(x)}$ tends to a limit $\varphi(t)$ as $x \to \infty$ for every t positive, then (i) $\varphi(t) = t^{\alpha}$, where α is a real number; (ii) the convergence of $\frac{f(tx)}{f(x)}$, as $x \to \infty$, is uniform in t on every interval $[a, b]$, $0 < a < b < \infty$; (iii) there exists X such that $\log f$ is bounded on every finite subinterval of $[X, \infty)$; and (iv) for $x > X$, f can be written in the form

$$f(x) = x^{\alpha} \exp \left[C(x) + \int_X^x \frac{\delta(t)}{t} dt\right],$$

where $C(x)$ and $\delta(x)$ are bounded measurable functions on $[X, \infty)$, convergent to zero as $x \to \infty$. With $g(x) = \log f(e^x)$, the fundamental result takes the following form: If g is a measurable real-valued function, and if

$$\Delta_t g(x) = g(x+t) - g(x) \to \psi(t) \tag{1}$$

as $x \to \infty$ for every real t, then (i) $\psi(t) = At$ with some real A; (ii) the convergence in (1) is uniform in t on bounded sets; (iii) there exists X such that $g(x)$ is bounded on every finite subinterval of $[X, \infty)$; and (iv) for $x > X$, $g(x) = Ax + c(x) + \int_X^x \varepsilon(t)\, dt$, where $c(x)$ and $\varepsilon(x)$ are bounded measurable functions on $[X, \infty)$, convergent to zero as $x \to \infty$.

In this article we shall generalize the preceding result by replacing in (1) the first-order difference $\Delta_t g(x)$ by the n^{th} order difference

$$\Delta_t^n g(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} g(x+kt);$$

in other words we shall consider here measurable functions g which satisfy the condition

(2) $$\Delta_t^n g(x) \to \psi(t), \quad x \to \infty,$$

for every real t. This condition is a natural one, and there are important functions which do not satisfy (1), but satisfy (2) for some value of n; such a function, for example, is $\log \Gamma(x)$.

Instead of considering the n^{th} order equidistant difference $\Delta_t^n g(x)$ we could consider the general n^{th} order difference

$$\Delta_{t_1} \Delta_{t_2} \dots \Delta_{t_n} g(x) =$$

$$= \sum_{\varepsilon_1=0}^{1} \sum_{\varepsilon_2=0}^{1} \dots \sum_{\varepsilon_n=0}^{1} (-1)^{n+\varepsilon_1+\dots+\varepsilon_n} g(x+\varepsilon_1 t_1 + \dots + \varepsilon_n t_n),$$

and so, instead of the condition (2) we could consider the apparently stronger condition

(3) $$\Delta_{t_1} \Delta_{t_2} \dots \Delta_{t_n} g(x) \to \chi(t_1, t_2, \dots, t_n), \quad x \to \infty,$$

for every point $\tau = (t_1, t_2, \dots, t_n) \in R^n$.

However, for an arbitrary, not necessarily measurable, complex-valued function g, the conditions (2) and (3) are equivalent. This fact is easy to verify in the case $n = 2$ owing to the identity

(4) $$\Delta_{t_1} \Delta_{t_2} = \frac{1}{2} \Delta_{t_1}^2 + \frac{1}{2} \Delta_{t_2}^2 - \frac{1}{2} T_{2t_2} \Delta_{t_1-t_2}^2,$$

(where the translation operator T_t is defined by $T_t g(x) = g(x+t)$, a notation we are going to use throughout this paper) or to the identity

(5) $$\Delta_{t_1} \Delta_{t_2} = \Delta_{(t_1+t_2)/2}^2 - T_{t_2} \Delta_{(t_1-t_2)/2}^2 .$$

To prove this fact for general n we need the following.

THEOREM 1. *For every $n \geq 1$ there exists a positive integer k, rational numbers C_j, $j = 1, 2, \dots k$, and $2k$ linear forms l_j and z_j, $j = 1, 2, \dots, k$, on R^n with integer coefficients such that*

(6) $$\Delta_{t_1} \Delta_{t_2} \dots \Delta_{t_n} = \sum_{j=1}^{k} C_j T_{z_j(\tau)} \Delta_{l_j(\tau)}^n ,$$

where $\tau = (t_1, t_2, \dots, t_n)$.

If in this theorem the words "integer coefficients" are replaced by "rational coefficients", a slightly weaker statement is obtained which will later be referred to as the weak form of Theorem 1. The weak form is obviously sufficient to deduce the equivalence of (2) and (3), but its proof does not seem to be much simpler. We might also observe that for $n = 2$ the smallest k for which an identity (6) exists is 3, but the smallest k for which a weak form of identity (6) exists is 2; see (4) and (5).

A statement weaker than the weak form of Theorem 1—but still strong enough to imply that (2) and (3) are equivalent—is the following:

In the algebra of finite linear combinations of translation operators, any product of n difference operators belongs to the ideal generated by the $n^{\underline{th}}$ powers of all difference operators.

Using the equivalence of (2) and (3) we shall prove the following generalization of the fundamental result on the regularly varying functions:

THEOREM 2. *Let F be a measurable complex-valued function on R. If*

(7) *$\Delta_t^n F(x)$ converges as $x \to \infty$ to a limit $\psi(t)$ for every real t, then*

(i) *there exists a complex number A such that*

(8) $$\Delta_{t_1} \Delta_{t_2} \dots \Delta_{t_n} F(x) \to A t_1 t_2 \dots t_n, \text{ as } x \to \infty,$$

$$\text{for every } \tau = (t_1, t_2, \dots, t_n) \in \mathrm{R}^n.$$

(*In particular, $\psi(t)$, defined in (7), is equal to At^n*).

(ii) *the convergence in (8) is uniform in τ on bounded subsets of R^n. (In particular, the convergence in (7) is uniform in t on bounded subsets of R).*

(iii) *there exists X such that F is bounded on every finite subinterval of $[X, \infty)$.*

(iv) *on the interval $[X, \infty)$, F can be represented in the form $F(x) = \frac{A}{n!} x^n + f_0(x) + f_1(x) + \dots + f_n(x)$, where f_0 is bounded and measurable and tends to zero as $x \to \infty$, and, for $j = 1, 2, \dots, n$, the $j^{\underline{th}}$ derivative of f_j is continuous and tends to zero as $x \to \infty$.*

Proof of Theorem 1. We shall write $z_j = \sum_i z_{ji} t_i$, $l_j = \sum_i l_{ji} t_i$, $e^{t_i x} = x_i$. Since the mapping $T_u \to e^{ux}$ introduces an isomorphism between the algebra of finite linear combinations of translation operators and the algebra of finite linear combinations of the functions e^{ux}, $(u \in R)$, and since

$\Delta_u^n = (T_u - T_0)^n$, it will be sufficient to show the existence of integers z_{ji}, l_{ji} and rationals C_j, $i = 1, 2, ..., n, j = 1, 2, ..., k$ such that

$$(9) \qquad \prod_{i=1}^{n} (x_i - 1) = \sum_{j=1}^{k} C_j \prod_{i=1}^{n} x_i^{z_{ji}} \left(\prod_{i=1}^{n} x_i^{l_{ji}} - 1 \right)^n .$$

This will follow immediately if we prove that in the ring of all polynomials in n indeterminates over the rational number field the polynomial $(x_1 - 1)(x_2 - 1) \dots (x_n - 1)$, multiplied by a suitable monomial $x_1^{\gamma_1} \dots x_n^{\gamma_n}$, belongs to the ideal J_n generated by all the polynomials of the form $(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} - x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n})^n$. (Here α_i, β_i, γ_i are non-negative integers and we assume $\alpha_i \beta_i = 0$, for $i = 1, 2, ..., n$). This is true for $n = 1$; in order to deduce the validity of the statement for n from its validity for $n - 1$, it is sufficient to show that the polynomial

$$(10) \qquad x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}} (x_n - 1) (x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} - x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}})^{n-1} =$$

$$= (x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}})^n (x_n - 1) (x_1^{\varepsilon_1} \dots x_{n-1}^{\varepsilon_{n-1}} - 1)^{n-1}$$

belongs to J_n. To show this and so to complete the proof it suffices to prove

(11) Let I_n denote the ideal generated by the polynomials $(x - 1)^n$, $(y - 1)^n$, $(y - x)^n$, $(y - x^2)^n$, ..., $(y - x^{2n-1})^n$ in the ring of all polynomials in x, y with rational coefficients. Then $(x - 1)^r (y - 1)^{n-r} \in I_n$ for $0 \leqq r \leqq n$,

then to apply (11) with $r = 1$, $x = x_n$, $y = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_{n-1}^{\varepsilon_{n-1}}$, and to remark that $\alpha_i \beta_i = 0$ for every i implies that if in (10) ε_i is negative, then $\varepsilon_i = -\beta_i$.

To prove (11) we denote by Z_n the set of all positive integers m such that $(x - 1)^r (y - 1)^{m-r} \in I_n$ for every integer r, $0 \leqq r \leqq m$, so that we have only to show $n \in Z_n$. Since obviously $2n - 1 \in Z_n$, it is sufficient to prove that

$$(12) \qquad m \in Z_n, \; 2n - 1 \geqq m > n, \; \text{implies} \; m - 1 \in Z_n.$$

Let s be an arbitrary integer such that $m - n \leqq s \leqq n - 1$ and let $D_k = D_k(m, n, s)$ satisfy

$$(13) \qquad \sum_{k=1}^{2n-m} D_k k^p = \delta_{ps}, \; m - n \leqq p \leqq n - 1,$$

where δ_{ps} is equal to 1 if $p = s$, to zero otherwise. (Such D_k exist, the determinant of the system (13) being different from zero).

With $P_k(x) = 1 + x + x^2 + \ldots + x^{k-1}$, $k = 1, 2, \ldots, 2n - m$, we have

$$(y - x^k)^{m-1} = [(y-1) - (x^k - 1)]^{m-1} =$$

$$= \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} (x-1)^p (y-1)^{m-1-p} P_k^p(x),$$

from which it follows that

$$\sum_{p=m-n}^{n-1} (-1)^p \binom{m-1}{p} (x-1)^p (y-1)^{m-1-p} P_k^p(x) \in I_n$$

for every $k = 1, 2, \ldots, 2n - m$. Multiplying these expressions respectively by $D_1, D_2, \ldots, D_{2n-m}$ and adding we find that

$$(14) \qquad \sum_{p=m-n}^{n-1} (x-1)^p (y-1)^{m-1-p} Q_p(x) \in I_n,$$

where $Q_p(x) = (-1)^p \binom{m-1}{p} \sum_{k=1}^{2n-m} D_k P_k^p(x)$.

Since $P_k^p(1) = k^p$, we obtain from (13) that for $p \neq s$ $Q_p(x)$ contains $x - 1$ as a factor, so that by the induction hypothesis (12) all the terms on the left-hand side of (14) with $p \neq s$ belong to I_n, so that (14) implies

$$(x-1)^s (y-1)^{m-1-s} Q_s(x) \in I_n.$$

Writing in the last formula $Q_s(x) = [Q_s(x) - Q_s(1)] + Q_s(1)$, noticing that $Q_s(x) - Q_s(1)$ contains $x - 1$ as a factor, using once more (12) and then observing that (13) implies $Q_s(1) \neq 0$, we obtain finally that

$$(x-1)^s (y-1)^{m-1-s} \in I_n$$

for $m - n \leqq s \leqq n - 1$ (and trivially so for $0 \leqq s < m - n$ and $n - 1 < s \leqq m - 1$), so that $m - 1 \in Z_n$, which was to be proved.

Remark 1. The ideal J_n, defined in the preceding proof, is given for every n by an infinite set of its generators, not by a basis; from this set of generators we can choose different bases for J_n, and to each basis there will correspond a formula of type (6). The real difficulty of the proof is to guess for each n a basis of J_n so that these bases for different n are connected in such a way that the induction step can be actually performed. Accordingly, the essential and the only non-trivial part of our proof is the statement and the proof of (11), especially our definition of the ideal I_n. (It can be deduced from (11) that the ideal I_n is symmetric in the sense that $P(x, y) \in I_n$ implies $P(y, x) \in I_n$. Rather peculiarly, the system of generators of I_n which we have used to define I_n and to prove the theorem is not symmetric in that sense).

We would like to stress the fact that, although Theorem 1 is stated as an existence theorem, its proof given above is constructive and can be used to find $k, C_j, l_{ji}, z_{ji}, j = 1, \ldots, k, i = 1, \ldots, n$, such that (6) holds.

Remark 2. We can express Theorem 1 in more intuitive terms in the following way. Let us call a rod any line-segment $[a, b]$ in R^n which carries at the point $z_j = \dfrac{(n-j)\,a + jb}{n}$ the electric charge $(-1)^j \binom{n}{j} c$, for every $j = 0, 1, \ldots, n$. Here c is a rational number which can vary from one rod to another. Let G be the lattice of all the points in R^n with integer coordinates, and C the system of 2^n electric charges, all of absolute value 1, situated at the vertices of the unit cube in R^n, in such a way that the charges at the endpoints of the same edge of the cube are of opposite sign and that the charge at the origin is positive. Theorem 1 is then equivalent to the following statement: it is possible to find finitely many rods such that each charge on each rod lies at some point of G and that when charges at the same point are added, the resulting non-zero sums form the system C. To verify this equivalence we assign to every expression of the form $Ax_1^{s_1} x_2^{s_2} \ldots x_n^{s_n}$, where $s_1, s_2, \ldots, s_n$ are integers, the electric charge A at the point $(s_1, s_2, \ldots s_n.)$ We then observe that in this way to the expression $(-1)^n (x_1 - 1) \ldots (x_n - 1)$ there corresponds the system C, and to the expression $C_j \prod_{i=1}^{n} x_i^{z_{ji}} \left(\prod_{i=1}^{n} x_i^{l_{ji}} - 1 \right)^n$, i.e. to any summand on the righthand side of (9), there corresponds a rod with charges situated at the points of G.

If in the given geometric interpretation we replace G by G^*, where G^* denotes the set of all the points in R^n with rational coordinates, we obtain a geometric interpretation of the weak form of Theorem 1.

We shall use the given geometric interpretation to describe in an intuitive way first an identity (6) of the weak form and then an identity (6), in the case $n = 3$. In both identities k will be 16 (we did not try to find out whether this is the smallest possible value for k); in other words in each case we shall use 16 rods to form the system C in R^3.

We take four rods of unit length having the same c and place them along the edges of the unit cube parallel to the x-axis in such a way that the charges at the vertices have appropriate signs. In this manner we obtain eight superfluous charges (superfluous means not lying at one of the vertices of the unit cube) which can be cancelled by four additional rods of length 1 parallel to the y-axis. The newly created eight superfluous electric charges can be cancelled out by using four rods of length 1 parallel to the z-axis. The superfluous charges created at the last stage lie at the vertices of the cube concentric with the unit cube and having edges equal to 1/3. To eliminate these charges we need four more rods, of length $\sqrt{3}$, situated along the four body diagonals of the cube. With the proper choice of the constant c, the system so obtained is identical to the system C. This construction is a geometric equivalent of an identity (6) of the weak form in the case $n = 3$.

If the preceding construction is slightly changed, namely if all the rods are chosen of length three times bigger, and if the first four rods are placed so that their middle thirds coincide with the edges of the unit cube parallel to the x-axis, we shall again obtain the system C. This new construction would be a geometric equivalent of an identity (6) in the case $n = 3$.

Remark 3. Let J_n have the same meaning as in the proof of Theorem 1. Then Theorem 1 is equivalent to the statement: there exists a monomial $x_1^{\gamma_1} \dots x_n^{\gamma_n}$ such that

$$x_1^{\gamma_1} \dots x_n^{\gamma_n} (x_1 - 1) \dots (x_n - 1) \in J_n .$$

The weak form of Theorem 1 is equivalent to the statement: there exists a positive integer m and a monomial $x_1^{\delta_1} \dots x_n^{\delta_n}$ such that

$$x_1^{\delta_1} \dots x_n^{\delta_n} (x_1^m - 1) \dots (x_n^m - 1) \in J_n .$$

Proof of Theorem 2.

(i) For $\tau = (t_1, t_2, \dots, t_n)$ we write $\Delta_\tau^{(n)} = \Delta_{t_1} \Delta_{t_2} \dots \Delta_{t_n}$.
As mentioned earlier, using Theorem 1 we deduce from (7) that

$$\Delta_\tau^{(n)} F(x) \to \chi(t_1, t_2, \dots, t_n) , \quad x \to \infty ,$$

for every $\tau \in R^n$. Since F is measurable, the function χ is measurable in each of the variables t_i separately, $i = 1, 2, \dots, n$. On the other hand, $\Delta_{a+b} = T_b \Delta_a + \Delta_b$ implies that

$$\chi(t_1, \dots, t_i' + t_i'', \dots, t_n) = \chi(t_1, \dots, t_i', \dots, t_n) + \chi(t_1, \dots, t_i'', \dots, t_n) .$$

Consequently, as a function of t_i, χ is a measurable solution of the Cauchy equation $h(x+y) = h(x) + h(y)$, which implies that

$$\chi(t_1, t_2, ..., t_n) = C_i(t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)\, t_i .$$

This being true for every i, $i = 1, 2, ..., n$, we obtain $\chi(t_1, t_2, ..., t_n) = A t_1 t_2 \ldots t_n$ for some constant $A = A_F$. This proves conclusion (i) of the theorem.

Writing

$$f(x) = F(x) - \frac{A}{n!} x^n , \tag{15}$$

and observing that $\Delta_\tau^{(n)} x^n = n!\, t_1 t_2 \ldots t_n$ for every x, we obtain

(16) $\Delta_\tau^{(n)} f(x) \to 0$, $x \to \infty$, for every $\tau \in R^n$.

In the remaining part of this proof we shall consider complex-valued measurable functions f satisfying (16), since from the results obtained in that case and from (15) the conclusions (ii)-(iv) of the theorem will follow immediately.

(ii) We shall use the following notation: $\varrho = (r_1, ..., r_n)$, $\sigma = (s_1, ..., s_n)$ $\tau = (t_1, ..., t_n)$ are points in R^n; if a is a positive number, Q_a is the cube $\{\tau \mid |t_i| \leqslant a \text{ for } i = 1, 2, ..., n\}$; d and ε are two arbitrary positive numbers; $\varepsilon' = 2^{-n}\varepsilon$; c is a positive number satisfying

$$2^n (c+d)^n - 2^n c^n + d^n < (c+d)^n ; \tag{17}$$

$b = c + d$; the letter S is used for arbitrary subsets of the set $K = \{1, 2, ..., n\}$; $|S|$ is the number of the elements in S; $\tau(S)$ is that point of R^n which has its $i^{\underline{th}}$ coordinate equal to t_i if $i \in S$ and to zero if $i \notin S$; $\lambda(\tau, S)$ denotes the expression $\sum_{i \in S} t_i$; f is a measurable complex-valued function satisfying (16); and

$$N(x, \varepsilon, d) = \{\tau \mid \tau \in Q_d,\ |\Delta_\tau^{(n)} f(x)| < \varepsilon\}$$

$$N_{\tau,S}(x, \varepsilon, c) = \tau(S) + N(x + \lambda(\tau, S), \varepsilon, c) .$$

In order to prove that the convergence in (16) is uniform in τ on bounded subsets of R^n, it is sufficient to show

(18) For any positive ε and d, there exists $X_{\varepsilon,d}$ such that $Q_d \subset N(x, \varepsilon, d)$ for $x > X_{\varepsilon,d}$.

For this purpose we need the following simple identity for difference operators

$$(19) \qquad \Delta_\tau^{(n)} = \sum_S (-1)^{|S|} T_{\lambda(\tau,S)} \Delta_{\varrho-\tau(S)}^{(n)}$$

and also the following result:

(20) For any positive ε and d there exists $X_{\varepsilon,d}$ such that $\bigcap_S N_{\tau,S}(x, \varepsilon', c)$ is non-empty for all $\tau \in Q_d$ and all $x > X_{\varepsilon,d}$.

To prove (19) we observe that

$$\Delta_{\rho+\sigma}^{(n)} = \prod_{i=1}^{n} \Delta_{r_i+s_i} = \prod_{i=1}^{n} (T_{r_i} \Delta_{s_i} + \Delta_{r_i})$$

implies

$$\Delta_{\rho+\sigma}^{(n)} = \sum_S T_{\lambda(\rho,S)} \prod_{i\in S} \Delta_{s_i} \prod_{i\in K-S} \Delta_{r_i} .$$

Substituting in the last equality $\sigma = \tau - \rho$ and noticing that $\Delta_{s_i} = T_{s_i} - T_0 = - T_{s_i} \Delta_{-s_i}$, we obtain (19).

To prove (20), we observe that (16) implies

$$(21) \qquad m(N(x, \varepsilon', c)) \to m(Q_c), \ x \to \infty .$$

For, if (21) is not true, there exists $\delta > 0$ and a sequence $\{ x_i \}$, $x_i \to \infty$, such that $m(N_i) < m(Q_c) - \delta$ for every $i = 1, 2, \ldots$; here $N_i = N(x_i, \varepsilon', c)$. This implies that the set $N = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} N_i$ has measure $< m(Q_c)$, so that there exists a point τ_0 in Q_c which does not belong to N. Thus $\tau_0 \notin N_i$ for infinitely many i, which is impossible, since, by (16), $\Delta_{\tau_0}^{(n)} f(x_i) \to 0$ as $x_i \to \infty$.

By (21), there exists $Y_{\varepsilon,d}$ such that $m(N(x, \varepsilon', c)) > m(Q_c) - d^n$ for $x > Y_{\varepsilon,d}$. Since also $\lambda(\tau, S) \geqslant - nd$ for $\tau \in Q_d$, we obtain that for every $\tau \in Q_d$ each of the 2^n sets $N_S = N_{\tau,S}(x, \varepsilon', c)$ will have measure $> m(Q_c) - d^n$ if $x > X_{\varepsilon,d} = Y_{\varepsilon,d} + nd$. Observing that each of the sets N_S is contained in Q_b, that $m(Q_c) = 2^n c^n$, $m(Q_b) = 2^n (c+d)^n$, and so, for every S, $m(Q_b - N_S) = m(Q_b) - m(N_S) < 2^n (c+d)^n - 2^n c^n + d^n$, and that the last expression is by (17) smaller than $(c+d)^n$, we have

$$m(\bigcup_S (Q_b - N_S)) \leq \sum_S m(Q_b - N_S) < 2^n (c+d)^n = m(Q_b) ,$$

which implies that $\bigcup_S (Q_b - N_S) \neq Q_b$. Taking complements with respect to Q_b of both sides in this inequality, we obtain (20).

Let τ be an arbitrary point of Q_d and let $x > X_{\varepsilon,d}$. We deduce from (20) that there exists $\rho = \rho(x, \tau)$ such that $\rho \in N_{\tau,S}(x, \varepsilon', c)$ for every S. This means that

$$|\Delta^{(n)}_{\rho-\tau(S)} f(x+\lambda(\tau, S))| < \varepsilon'$$

for every S, and implies that every summand on the right-hand side of (19) is in absolute value smaller than ε', so that, by (19), $|\Delta^{(n)}_{\tau} f(x)| < 2^n\varepsilon' = \varepsilon$, which gives (18).

(iii) By the observation made at the end of part (i), and by the result of part (ii) of this proof, in order to prove the conclusion (iii) of the theorem it is sufficient to establish

(22) If f is a complex-valued measurable function on R, such that $\Delta^n_t f(x) \to 0$, $x \to \infty$, uniformly in t on the interval $[0, 1]$, then there exists X such that f is bounded on every finite subinterval of $[X, \infty)$.

Let us assume that the conclusion of (22) does not hold. Then there exists a sequence $\{x_m\}$, $x_m \to \infty$, such that f is unbounded on each of the intervals $I_m = (x_m, x_m+1)$. This implies that for every m there exists a sequence of points $y_{m1}, y_{m2}, \ldots, y_{mn}, \ldots$ in I_m such that $y_{mn} \to y_{m0}$, $|f(y_{mn})| \to \infty$ as $n \to \infty$.

For $r = 1, 2, \ldots, n$, write

$$S_{m,k,r} = \{x \mid -1 < x - y_{m0} < r+1\,,\ |f(x)| < k\}\,.$$

Then $m(S_{m,k,r}) \to r+2$ as $k \to \infty$, so that for every m there exists $k = k(m)$ such that

$$m(S_{m,k,r}) > r + 2 - \frac{r}{2n} \quad \textit{for} \quad r = 1, 2, \ldots, n\,. \tag{23}$$

Let $j = j(m)$ be such that $|f(y_{mj})| > m + 2^m k$, and let $I_{m,r}$ denote the interval $(y_{mj}, y_{mj}+r)$. It follows easily from (23) that

$$m(I_{m,r} \cap S_{m,k,r}) > r - \frac{r}{2n} \quad \textit{for} \quad r = 1, 2, \ldots, n\,.$$

Let

$$U_{m,r} = \{t \mid 0 < t < 1\,,\ |f(y_{mj}+rt)| < k\}\,.$$

Then

$$y_{mj} + r U_{m,r} = I_{m,r} \cap S_{m,k,r}\,,$$

so that $m(U_{m,r}) > 1 - \dfrac{1}{2n}$ for $r = 1, 2, \ldots, n$. Thus the n sets $U_{m,r}$, $r=1, 2, \ldots, n$, which are all contained in $(0,1)$, have a point t_m in common.

We have then $0 < t_m < 1$, $|f(y_{mj})| > m + 2^m k$ and $|f(y_{mj}+rt_m)| < k$ for $r = 1, 2, \ldots, n$ which gives

$$|\Delta^n_{t_m} f(y_{mj})| \geq |f(y_{mj})| - \sum_{r=1}^{n} \binom{n}{r} |f(y_{mj}+rt_m)| >$$

$$> m + 2^m k - 2^m k = m$$

for $m = 1, 2, \ldots$; and this contradicts the assumption of (22).

(iv) Let f be a measurable complex valued function satisfying (16). Then, by the conclusion (iii) of the theorem, there exists X such that f is locally integrable on $[X, \infty)$, and so $Mf(x) = \int_x^{x+1} f(t)\,dt = \int_0^1 f(x+t)\,dt$ is defined for $x > X$. The opeiators M and $M - I$, where I is the identity operator, commute, so that.

$$I = [M - (M-I)]^{n+1} = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} M^{n+1-k}(M-I)^k$$

From this it follows that for $x > X$

$$f(x) = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} h_k(x)\,, \tag{23}$$

where $h_k(x) = M^{n+1-k}(M-I)^k f(x)$ for $k = 0, 1, \ldots, n+1$. We shall show that

$$D^{n-k} h_k(x) \text{ is continuous for } x > X\,,\ k = 0, 1, \ldots, n\,, \tag{24}$$

$$D^{n-k} h_k(x) \to 0\,,\ x \to \infty \text{ for } k = 0, 1, \ldots, n \tag{25}$$

$$h_{n+1}(x) \to 0\,,\ x \to \infty\,, \tag{26}$$

where $D^j h$ denotes the j^{th} derivative of h.

From these three facts, writing $f_0 = (-1)^{n+1} h_{n+1} + (-1)^n \binom{n+1}{n} h_n$ and $f_j = (-1)^{n-j} \binom{n+1}{n-j} h_{n-j}$ for $j = 1, 2, \ldots, n$, and using (15) and (16), we can easily deduce the conclusion (iv) of the theorem. (The boundedness of the function f_0 on $[X, \infty)$ will follow from the fact that $h_n(x) \to 0$, $h_{n+1}(x) \to 0$ as $n \to \infty$, from the conclusion (iii) of the theorem, the fact that the functions $h_j(x)$, $j = 0, \ldots, n$ are continuous on $[X, \infty)$ and the equality (23)).

We observe that for $x > X$, $Mf(x)$ is continuous, and so $DM(Mf)(x) = Mf(x+1) - Mf(x) = \Delta_1 Mf(x)$. From this we deduce that for $k = 0, 1, \ldots, n$

$$D^{n-k} h_k = D^{n-k} M^{n+1-k}(M-I)^k f = \Delta_1^{n-k}(M-I)^k Mf. \tag{27}$$

Since $Mf(x)$ is continuous for $x > X$, the right-hand side of (27) is continuous on $[X, \infty)$, and so (24) follows from (27).

For any function g locally integrable on $[X, \infty)$, $(M-I)\,g(x) = \int_0^1 (g(u+x) - g(x))\,du = \int_0^1 \Delta_u g(x)\,du$, and it follows by induction that, for $k = 1, 2, \ldots$,

$$(28) \qquad (M-I)^k g(x) = \int_0^1 \ldots \int_0^1 \Delta_{u_1} \ldots \Delta_{u_k} g(x)\,du_1 \ldots du_k\,.$$

By (27) and (28) we obtain

$$(29) \qquad D^{n-k} h_k = \int_0^1 \ldots \int_0^1 \Delta_{u_1} \ldots \Delta_{u_k} \underbrace{\Delta_1 \ldots \Delta_1}_{n-k} Mf(x)\,du_1 \ldots du_k$$

Since (16) implies that $\Delta_\tau^{(n)} Mf(x) \to 0$, $x \to \infty$ for every $\tau \in R^n$, we see, by applying the conclusion (ii) of the theorem to the function Mf, that the integrands in the last integral converge uniformly to zero as $x \to \infty$, so that from (29) follows (25).

Since $\Delta_\tau^{(n)} f(x) \to 0$, $x \to \infty$ for every $\tau \in R^n$ implies $\Delta^{(n+1)} f(x) \to 0$, $x \to \infty$ for every $\tau \in R^{n+1}$, using (28) with $k = n+1$, $g = f$, and applying the conclusion (ii) of the theorem with n replaced by $n+1$, we obtain from (28) that (26) holds.

Related to Theorem 2 is the following

THEOREM 3. *Let f be a complex valued measurable function on R satisfying*

$$(30) \quad \limsup_{x \to \infty} \left| \Delta_t^n f(x) \right| < \infty \textit{ for every } t \in R.$$

Then for every bounded subset B of R^n there exist K_B and X_B such that

$$| \Delta_{t_1} \ldots \Delta_{t_n} f(x) | < K_B$$

for $x > X_B$ and $(t_1, t_2, \ldots, t_n) \in B$.

In the case $n = 1$ Theorem 3 was proved by I. Csiszár and P. Erdös [4].

Proof of Theorem 3. By Theorem 1, (30) implies

$$(31) \quad \limsup_{x \to \infty} \left| \Delta_\tau^{(n)} f(x) \right| < \infty, \text{ for every } \tau \in R^n.$$

Using the notation introduced in part (ii) of the proof of Theorem 2, we see that it is sufficient to show

(18′) For any d positive, there exist X_d and ε_d such that $Q_d \subset N(x, \varepsilon_d, d)$ for $x > X_d$.

For that purpose we have only to show

(20′) For any d positive there exist ε_d' and X_d such that $\bigcap_s N_{\tau,s}(x, \varepsilon_d', c)$ is non-empty for $\tau \in Q_d$ and $x > X_d$,

since in the same way in which we have deduced (18) from (19) and (20) we can deduce (18′) from (19) and (20′), taking $\varepsilon_d = 2^n \varepsilon_d'$.

Investigating the proof of (20), we see that in order to prove (20′) we need only

(32) For every d positive there exist ε_d' and Y_d such that

$$m\left(N(x, \varepsilon_d', c)\right) > m(Q_c) - d^n \quad for \quad x > Y_d .$$

Let us assume that (32) does not hold. Then there exists a sequence $\{ x_k \}$, $x_k \to \infty$ such that

$$m\left(N(x_k, k, c)\right) \leq m(Q_c) - d^n$$

for $k = 1, 2, \ldots, n$. It follows that the set $N = \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} N_k$, where $N_k = N(x_k, k, c)$, is a proper subset of Q_c. So there exists $\tau_0 \in Q_c$ such that $\tau_0 \notin N_k$ for infinitely many k. This means that $\left| \Delta_{\tau_0}^{(n)} f(x_k) \right| \geqslant k$ for infinitely many k, which is in contradiction with (31).

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