# §1. Banach vector bundles over an analytic space

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

*Remark :* This a particular case of the following proposition: if $\pi$ and $\pi'$ are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ & {}_{\pi}\searrow \; S \; \swarrow_{\pi'} & \end{array} \qquad \mathcal{O}_{X \underset{S}{\times} Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_y .$$

We have proved that $\mathcal{O}_{W \times X}$ is $\mathcal{O}_W$-flat, so by scalar extension $\mathcal{O}_{S \times X}$ is $\mathcal{O}_S$ flat.

*Corollary :* If $X$ and $S$ are two manifolds and $\pi : X \to S$ is a submersion, then $\pi$ is flat.

## III. Privileged polycylinders

### § 1. *Banach vector bundles over an analytic space*

Let $E$ be a Banach space and $X$ an analytic space. We denote then by $E_X$ the trivial bundle $X \times E$ over $X$.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from $X$ to $E$. If $U \subset \mathbf{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from $U$ into $E$ consists of all functions $g : U \to E$ having at every point $x \in U$ a converging power series expansion.

Let now $X'$ be a local model for $X$, i.e. $X'$ is the support of the quotient sheaf $\mathcal{O}_U / J$, where $U \subset \mathbf{C}^n$ is open and $J$ is a coherent sheaf of ideals of $\mathcal{O}_U$, then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \to \mathcal{H}(V, E) / J_V \,.\, \mathcal{H}(V, E)$ ($V \subset U$, $V$-open).

*Remark :* If $X'$ is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from $X'$ to $E$ which are locally induced by analytic functions on open sets in $U$.

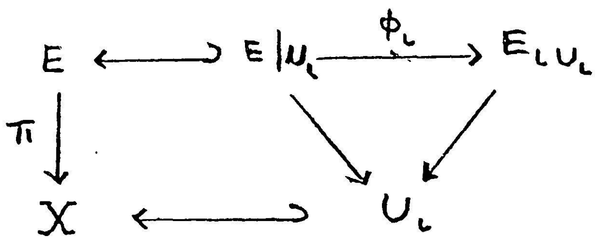
The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models $X'$ of $X$, i.e. $\mathcal{H}_X(E) | X' = \mathcal{H}_{X'}(E)$, for every local model $X'$.

*Definition 1 :* The set of *analytic morphisms* from an analytic space $X$ into a Banach space $E$ is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space $E$ into the Banach space $F$.

*Definition 2 : An analytic vector bundle morphism* from $E_X$ into $F_X$ is an analytic morphism from $X$ into $\mathcal{L}(E, F)$.

Let $E$ be a topological space, $X$ an analytic space, and $\pi : E \to X$ a continuous projection.




Suppose that $X$ has an open covering $(U_\iota)_{\iota \varepsilon I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E_{\iota U_\iota}$ and a homeomosphism $\phi_\iota$, such that the following diagram is commutative:

We suppose further that for each pair $\iota, \kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa} : E_{\kappa U_\iota \cap U_\kappa} \to E_{\iota U_\iota \cap U_\kappa}$, with the underlying mapping $\phi_\iota \circ \phi_\kappa^{-1}$, such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\,\gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \qquad \text{for all} \quad \iota, \kappa, \gamma \in I .$$

This data gives a Banach vector bundle atlas on $E$ and provides $E$ with the structure of a Banach vector bundle over $X$ (two atlases are equivalent if there exists an atlas containing both).

*Remark:* If $X$ is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\,\gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

*Proposition 1:* Let $\phi : E \to F$ (over $X$) be a morphism of two Banach vector bundles $E$ and $F$, and $x \in X$.

If $\phi_x \in \mathscr{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of $x$, such that $\phi|U : E|U \to F|U$ is a vector bundle isomorphism.

*Proof:* First we take a trivialisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x \in V \subset X$ ($V$-open).

The set Isom $(E_0, F_0)$ of isomorphic mappings is an open subset of $\mathscr{L}(E_0, F_0)$ and the mapping $g \to q^{-1}$ is an analytic isomorphism:

$$\text{Isom}(E_0, F_0) \simeq \text{Isom}(F_0, E_0) .$$

So we have in an open neighbourhood $U \subset X$ of $x$ an analytic morphism $y \to \phi_y^{-1} \in \mathscr{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|U)^{-1} : F|U \to E|U$.

*Definition 3 :* Let $E$ and $F$ be two Banach spaces and $f$ a continuous linear mapping from $E$ into $F$. $f$ is a *split mono-(epi) morphism,* if there exists a mapping $g \in \mathscr{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

*Definirion 4 :* Let $E_1$ and $E_2$ be two Banach vector bundles over an analytic space $X$, and $f$ a vector bundle morphism from $E_1$ into $E_2$. $f$ *is a split mono (epi) morphism,* if there exists a vector bundle morphism $g : E_2 \to E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, $f : E_1 \to E_2$ (over $X$) is a split monomorphism if an only if $E_2$ can

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \to G_2 \end{cases}.$$

and $f$ is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that} \quad f: \begin{cases} F_1 \to 0 \\ G_1 \simeq E_2 \end{cases}.$$

*Proposition 2 :* Let $E \xrightarrow{\varphi} F$ be a bundle morphism and $x \in X$.

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

If $\phi_x : E(x) \to F(x)$ is a split epi (mono) morphism, then the point $x$ has an open neighbourhood $U \subset X$, such that $\phi | U : E | U \to F | U$ is a split vector bundle epi (mono) morphism.

*Proof :* Suppose that $\phi_x$ is a split epimorphism. We take first a trivilisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x$, so that there exists a mapping $\sigma \in \mathscr{L}(F_0, E_0)$, $\phi_x \circ \sigma = I_{F_0}$. If we define a morphism $\psi : F_{0V} \to E_{0V}$ by $x \to \sigma \in \mathscr{L}(F_0, E_0)$, the morphism $\gamma = \phi \circ \psi : F_{0V} \to F_{0V}$ has an isomorphic fibre mapping $\gamma_x = I_{F_0}$ in $x$. By proposition 1 we have an isomorphic restriction $\gamma | U$, $\phi | U \circ (\psi | U \circ (\gamma | U)^{-1}) = I_{F_{0U}}$.

When $\phi_x$ is a split monomorphism, the proof is similar.

*Definition 5 :* Let $B_1$, $B_2$, $B_3$ be Banach spaces, and $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$ continuous linear mappings. This sequence forms *a complex,* if $k \circ j = 0$. This sequence is *split exact* if the space $B_i$ can be decomposed in direct sums $B_i = C_i \oplus D_i$ such that

$$j: \begin{cases} C_1 \to 0 \\ D_1 \simeq C_2 \end{cases} \qquad k: \begin{cases} C_2 \to 0 \\ D_2 \simeq C_3 \end{cases}.$$

*Definition 6:* A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

is *a complex* if $g \circ f = 0$.

The sequence is *split exact*, if every $E_i$ can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f: \begin{cases} F_1 \to 0 \\ G_1 \simeq F_2 \end{cases} \qquad g: \begin{cases} F_2 \to 0 \\ G_2 \simeq F_3 \end{cases}.$$

*Theorem 1:* Let $\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$ be a complex of Banach vector bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \overset{fx_0}{\to} E_2(x_0) \overset{fx_0}{\to} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of $x_0$, such that $E_1|U \overset{f|U}{\to} E_2|U \overset{g|U}{\to} E_3|U$ is a split exact sequence of Banach vector bundles.

*Proof:* We take a neighbourhood $V$ of $x$, such that we have a complex $E_{1V} \overset{f|V}{\to} E_{2V} \overset{g|V}{\to} E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0}: \begin{cases} F_1(x_0) \to 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \qquad g_{x_0}: \begin{cases} F_2(x_0) \to 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases}.$$

By proposition 2, $f|V: G_{1V} \to E_{2V}$, $g|V: G_{2V} \to E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of $x_0$ and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of $E_{2W}$ esp. $E_{3W}$, such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W}.$$

By our construction

$$g \mid W : \begin{cases} F_2 & \to 0 \\ G_2 W \simeq F_3 \end{cases} .$$

If $p: E_{2W} \to F_2$ is the projection with kernel $G_{2W}$, the map, $p \circ f: E_{1W} \to F_2$ is a split epimorphism in $x_0$. Again by prop. 2 we have over an open eighbourhood $U \subset W$ of $x_0$ a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \operatorname{Ker} p \circ f$)

$$(p \circ f) \mid U : \begin{cases} F_1 & \to 0 \\ G_{1U} & \tilde{\to} F_{2U} \end{cases} .$$

The image $f|U(F_1)$ is contained in $G_{2U}$. But $g|U \circ f|U = 0$ and $g|G_{2U}$ is a monomorphism hence $f|U : F_1 \to 0$. We get finally (restricting all our morphisms to $U$)

$$f \mid U : \begin{cases} F_{1U} \to 0 \\ G_{1U} \simeq F_{2U} \end{cases} \qquad g \mid U : \begin{cases} F_{2U} \to 0 \\ G_{2U} \tilde{\to} F_{3U} \end{cases} .$$

## § 2. *Privileged polycylinders*

*Definition 1: A polycylinder* in $\mathbf{C}^n$ is a compact set $K$ of the form $K = K_1 \times \ldots \times K_n$ where each $K_i$ is a compact, convex subset of $\mathbf{C}$, with nonempty interior. If each $K_i$ is a disc, then $K$ is a polydisc. We first recall the following theorem of Cartan.

*Theorem 1:* Let $K$ be a polycylinder contained in an open subset $U$ of $\mathbf{C}^n$. Let $\mathscr{F}$ be a coherent analytic sheaf on $U$.

(A) There exists an open neighbourhood of $K$ over which $\mathscr{F}$ admits a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0 .$$

(B) $H^q(K, \mathscr{F}) = 0$ for $q > 0$.
(Reference: For instance Gunning and Rossi.)
We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

of a coherent sheaf $\mathscr{F}$, the sequence