

# §3. Introduction to flatness by examples

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

*Remark* : The inverse image is a particular case of the analytic pull-back.

In fact, if  $Y$  is a closed analytic subspace of  $X$  and  $f : X' \rightarrow X$  is a morphism:

$$f^* \mathcal{O}_Y = f_0^* (\mathcal{O}_X / J_Y) \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'}$$

$$\simeq \mathcal{O}_{X'} / f^1(J_Y) \cdot \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that  $A/I \otimes_A E \simeq E/IE$ ).

*Elementary properties of the analytic pull-back :*

- (a)  $(f^* \mathcal{E})_{x'} = (f_0^* \mathcal{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$  where  $x = f_0(x')$  (since  $\otimes$  commutes with inductive limits).
- (b)  $f^* (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f^* \mathcal{E} \otimes_{\mathcal{O}_{X'}} f^* \mathcal{F}$ , where  $\mathcal{E}$  and  $\mathcal{F}$  are  $\mathcal{O}_X$ -modules.
- (c) If  $\mathcal{E}$  is a coherent  $\mathcal{O}_X$ -module, then  $f^* \mathcal{E}$  is a coherent  $\mathcal{O}_{X'}$ -module.

In fact,  $\mathcal{E}$  has a locally finite presentation:

$$\mathcal{O}_X^q \rightarrow \mathcal{O}_X^p \rightarrow \mathcal{E} \rightarrow 0, \text{ and } f^* \text{ is compatible with cokernels, } f^* (\mathcal{O}_X^r) = \mathcal{O}_{X'}^r.$$

*Special case* : The pull-back of vector bundle. Let  $(E, \pi)$  be an analytic

$$\begin{array}{ccc} E \times X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

vector bundle over the analytic space  $X$ , and  $f : X' \rightarrow X$  a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over  $X'$ , such that  $\bar{f}$  is a bundle morphism. We call this bundle  $E'$ .

*Proposition 1* : Let  $\mathcal{E}$  (Resp.  $\mathcal{E}'$ ) be the sheaf of analytic sections of  $E$  (Resp.  $E'$ ). Then  $\mathcal{E}' = f^* \mathcal{E}$ .

*Proof (Sketch)* : We have a  $f_0^* \mathcal{O}_X$  linear morphism  $f_0^* \mathcal{E} \rightarrow \mathcal{E}'$ , which extends to a morphism  $f^* \mathcal{E} \rightarrow \mathcal{E}'$ . We can prove that this is an isomorphism. Since the question is local with respect to  $X'$ , we can suppose that  $E$  is a trivial bundle over  $X$  with fiber  $\mathbf{C}^r$ , then  $\mathcal{E} = \mathcal{O}_X^r$ . Also  $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$ . Therefore  $f^* \mathcal{E} = \mathcal{E}'$ .

### § 3. Introduction to flatness by examples

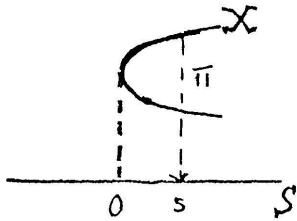
Let  $S$  be an analytic space. By analytic space over  $s$  we mean an analytic space  $X$  provided with a morphism  $\pi : X \rightarrow S$ . Let  $S$  be a simple point in  $S$ , and consider  $X(s) = f^{-1}(s)$ .

The main purpose of these lectures is to give a precise meaning to the expression:

“  $X(s)$  depends nicely on  $s$ ”, and to give a criterion for the “ nice ” behaviour.

We begin with some examples.

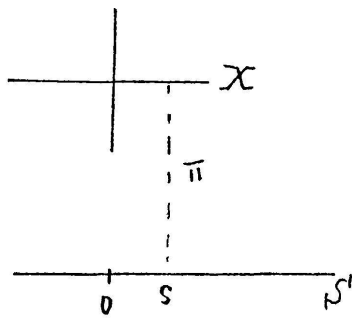
*Example 1:*  $X$  is the closed subspace on  $\mathbf{C}^2$  defined by  $(y^2 - x)$ ,  $S = \mathbf{C}$  and  $\pi = 1st$  projection.



$$X(s) = \begin{cases} 2 \text{ simple points if } s \neq 0 \\ \text{double point if } s = 0. \end{cases}$$

Here the behaviour of  $X(s)$  is nice.

*Example 2:*  $X$  is the closed subspace of  $\mathbf{C}^2$  defined by  $(xy)$ ,  $S = \mathbf{C}$  and  $\pi = 1st$  projection.



$X(s)$  is given by  $(x-s, xy)$ , and

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if } s \neq 0 \\ (x) & \text{if } s = 0. \end{cases}$$

The first case is a simple point, the second one the  $y$ -axis.

A similar example is the map of a point into  $\mathbf{C}$ .

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreducible component of  $X$ , and after removing this component  $\pi$  behaves nicely.

This kind of removing is not possible in general, as the following example shows:

*Example 3:*  $X$  is given in  $\mathbf{C}^3$  by  $(xz - y)$ , and  $\pi$  is the projection on the  $(x, y)$ -plane.

If  $s = (x_0, y_0)$ , then the fiber  $X(s)$  is defined by

$$(x-x_0, y-y_0, xz-y) = \begin{cases} \left(x-x_0, y-y_0, z-\frac{y_0}{x_0}\right) & \text{if } x_0 \neq 0 \\ (x, y) & \text{if } x_0 = y_0 = 0 \\ (1) & \text{if } x_0 = 0, y_0 \neq 0. \end{cases}$$

The set of “ nice ” fibers is dense in  $X$ , so we cannot remove the  $z$ -axis and still get a closed subspace of  $\mathbf{C}_3$ .

#### § 4. Algebraic study of flatness

In the following all rings are commutative, with 1, and all modules are unitary.

*Definition 1:* An  $A$ -module  $E$  is *flat*, if for every exact sequence of  $A$ -modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0,$$

the sequence  $0 \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$  is also exact. We can also say, because  $\otimes$  is right exact, that  $E$  is flat, if for every injective homomorphism  $F' \rightarrow F$ ,  $E \otimes F' \rightarrow E \otimes F$  is also injective.

*Examples of modules which are not flat :*

- (1) if  $A = \mathbf{Z}$ ,  $E = \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$ ,  $F = F' = \mathbf{Z}$ ; then the sequence  $0 \rightarrow \mathbf{Z} \xrightarrow{2I} \mathbf{Z} (2I : x' \rightarrow 2x)$  is exact. But now  $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$ , and the homomorphism  $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$  is the zero homomorphism, which is not injective. So  $\mathbf{Z}_2$  is not a flat  $\mathbf{Z}$  module.
- (2) If  $A = \mathbf{C}\{x\}$ ,  $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$ ,  $F = F' = \mathbf{C}\{x\}$ , then the sequence  $0 \rightarrow F \xrightarrow{xI} F' (xI : p(x) \rightarrow xp(x))$  is exact. But the homomorphism  $E \xrightarrow{xI} E$  is not injective.

*Proposition 1 :* If  $A$  is an integral domain and  $E$  a flat  $A$ -module, then  $E$  is torsion-free.

*Proof :* Let  $a \in A$ ,  $a \neq 0$ . Because  $A$  is an integral domain, the sequence  $0 \rightarrow AA \xrightarrow{aI} (aI : x \rightarrow ax)$  is exact. Since  $E$  is flat, the sequence  $0 \rightarrow E \xrightarrow{aI} E$  is also exact. In other words  $E$  has no torsion elements.

*Proposition 2 :* If  $A$  is a principal-ideal domain, then  $E$  is flat if and only if  $E$  is torsionfree.

*Proof :* See corollary of prop. 6.

*Examples of flat modules :*

- (1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.