§3. Introduction to flatness by examples

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Remark : The inverse image is a particular case of the analytic pull-back.

In fact, if Y is a closed analytic subspace of X and $f : X' \to X$ is a morphism:

$$f^* \mathcal{O}_Y = f_0^*(\mathcal{O}_X/J_Y) \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'}$$

$$\simeq \mathcal{O}_{X'}/f^1(J_Y).\, \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that $A/I \underset{A}{\otimes} E \simeq E/IE$).

Elementary properties of the analytic pull-back :

(*a*) $(f^* \mathscr{E})_{x'} = (f_0^* \mathscr{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathscr{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ where $x = f_0(x')$ (since $\otimes$ commutes with inductive limits).

(*b*) $f^* (\mathscr{E} \underset{\mathcal{O}X}{\otimes} \mathscr{F}) = f^* \mathscr{E} \underset{\mathcal{O}X'}{\otimes} f^* \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are $\mathcal{O}_X$-modules.

(*c*) If $\mathscr{E}$ is a coherent $\mathcal{O}_X$-module, then $f^* \mathscr{E}$ is a coherent $\mathcal{O}_{X'}$-module.

In fact, $\mathscr{E}$ has a locally finite presentation: $\mathcal{O}_X^q \to \mathcal{O}_X^p \to \mathscr{E} \to 0$, and f^* is compatible with cokernels, $f^* (\mathcal{O}_X^r) = \mathcal{O}_X^r$.

Special case : The pull-back of vector bundle. Let (E, π) be an analytic vector bundle over the analytic space X, and f: $X' \to X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over X', such that $\bar{f}$ is a bundle morphism. We call this bundle E'.

$$\begin{array}{ccc} E \underset{X}{\times} X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

Proposition 1 : Let $\mathscr{E}$ (Resp. $\mathscr{E}'$) be the sheaf of analytic sections of E (Resp. E'). Then $\mathscr{E}' = f^* \mathscr{E}$.

Proof (Sketch) : We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathscr{E} \to \mathscr{E}'$, which extends to a morphism $f^* \mathscr{E} \to \mathscr{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to X', we can suppose that E is a trivial bundle over X with fiber $\mathbf{C}^r$, then $\mathscr{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathscr{E} = \mathscr{E}'$.

§ 3. *Introduction to flatness by examples*

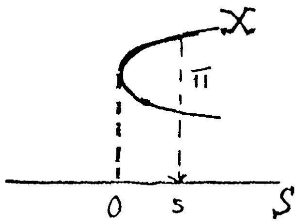
Let S be an analytic space. By analytic space over s we mean an analytic space X provided with a morphism $\pi : X \to S$. Let S be a simple point in S, and consider $X(s) = f^{-1}(s)$.

The main purpose of these lectures is to give a precise meaning to the expression:

" $X(s)$ depends nicely on s", and to give a criterion for the "nice" behaviour.

We begin with some examples.

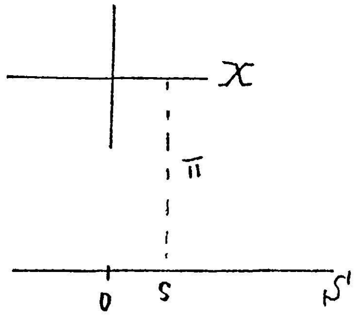
Example 1: X is the closed subspace on $\mathbf{C}^2$ defined by (y^2-x), $S = \mathbf{C}$ and π = 1st projection.



$$X(s) = \begin{cases} \text{2 simple points if } s \neq 0 \\ \text{double point if } \quad s = 0\,. \end{cases}$$

Here the behaviour of $X(s)$ is nice.

Example 2: X is the closed subspace of $\mathbf{C}^2$ defined by (xy), $S = \mathbf{C}$ and π = 1st projection.



$X(s)$ is given by $(x-s, xy)$, and

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if} \quad s \neq 0 \\ \quad (x) & \text{if} \quad s = 0\,. \end{cases}$$

The first case is a simple point, the second one the, y-axis.

A similar example is the map of a point into $\mathbf{C}$.

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreductible component of X, and after removing this component π behaves nicely.

This kind of removing is not possible in general, as the following example shows:

Example 3: X is given in $\mathbf{C}^3$ by $(xz-y)$, and π is the projection on the (x, y)-plane.

If $s = (x_0, y_0)$, then the fiber $X(s)$ is defined by

$$(x-x_0, y-y_0, xz-y) = \begin{cases} \left(x-x_0, y-y_0, z-\dfrac{y_0}{x_0}\right) \text{ if } x_0 \neq 0 \\ (x, y) \qquad\qquad\qquad\qquad \text{if} \quad x_0 = y_0 = 0 \\ (1) \quad \text{if} \quad x_0 = 0\; y_0 \neq 0\,. \end{cases}$$

The set of " nice " fibers is dense in X, so we cannot remove the z-axis and still get a closed subspace of $\mathbf{C}_3$.

§ 4. *Algebraic study of flatness*

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1: An A-module E is *flat*, if for every exact sequence of A-modules

$$0 \to F' \to F \to F'' \to 0\,,$$

the sequence $0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is also exact. We can also say, because $\otimes$ is right exact, that E is flat, if for every injective homomorphism $F' \to F$, $E \otimes F' \to E \otimes F$ is also injective.

Examples of modules which are not flat:

(1) if $A = \mathbf{Z}$, $E = \mathbf{Z}_2 = \mathbf{Z}/2\,\mathbf{Z}$, $F = F' = \mathbf{Z}$; then the sequence $0 \to \mathbf{Z} \overset{2I}{\to} \mathbf{Z}$ $(2I : x' \to 2x)$ is exact. But now $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$, and the homomorphism $\mathbf{Z}_2 \overset{2I}{\to} \mathbf{Z}_2$ is the zero homomorphism, which is not injective. So $\mathbf{Z}_2$ is not a flat $\mathbf{Z}$ module.

(2) If $A = \mathbf{C}\,\{x\}$, $E = \mathbf{C} = \mathbf{C}\,\{x\}/(x)$, $F = F' = \mathbf{C}\,\{x\}$, then the sequence $0 \to F \overset{xI}{\to} F'$ $(xI : p\,(x) \to xp\,(x))$ is exact. But the homomorphism $E \overset{xI}{\to} E$ is not injective.

Proposition 1: If A is an integral domain and E a flat A-module, then E is torsion-free.

Proof: Let $a \in A$, $a \neq 0$. Because A is an integral domain, the sequence $0 \to AA \overset{aI}{\to}$ $(aI : x \to ax)$ is exact. Since E is flat, the sequence $0 \to E \overset{aI}{\to} E$ is also exact. In other words E has no torsion elements.

Proposition 2: If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof: See corollary of prop. 6.

Examples of flat modules:

(1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.