Appendix Local noetherian rings

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Appendix

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All rings here as well as in the preceding lectures are supposed to be commutative and have units. A ring A is called *local* if it contains exactly one maximal ideal ; this will be denoted by $\mathfrak{m}(A)$ or simply $\mathfrak{m}$. A module E over a ring A is called *finite* if it is finitely generated over A. A module E over A is called *noetherian* if E is unitary (i.e. $1\,x = x$ for all $x \in E$) and every submodule of E is finite over A. In particular A itself is noetherian if and only if all its ideals are finitely generated.

We state without proof the following result.

Theorem (Lemma of Artin-Rees). Let A be a noetherian ring and I an ideal of A. Let E be a finite A-module and F_1, F_2 submodules of E. Then there is an integer n such that

$$I\left((I^n F_1) \cap F_2\right) = (I^{n+1} F_1) \cap F_2 \,.$$

The proof may be found in Nagata [8, Theorem 3.7] or, for $F_1 = E$ which is the only case we shall need, in Bourbaki [1, Ch. III, 3, no 1].

Lemma (called Nakayama's lemma by Bourbaki). Let A be a local ring with maximal ideal $\mathfrak{m}$ and E a finite A-module.

(i) If $E = \mathfrak{m}\,E$, then $E = 0$.

(ii) If F is a submodule of E such that $E = F + \mathfrak{m}\,E$, then $F = E$.

(iii) Let $k = A/\mathfrak{m}$, a field. Then $k \otimes_A E = 0$ implies $E = 0$.

Proof. (i) Let $x_1, \ldots, x_n$ be generators for E, $n \geqslant 1$. We can then write $x_n = \sum^n_1 a_j x_j$ for some $a_j \in \mathfrak{m}$, hence $(1 - a_n)\, x_n = \sum_1^{n-1} a_j x_j$. Since $1 - a_n$ is invertible this means that $x_1, \ldots, x_{n-1}$ generate E. The minimal number of generators must therefore be zero, i.e. $E = 0$.

(ii) We only need to apply (i) to E/F.

(iii) We have $k \otimes_A E = E/\mathfrak{m}E$ which reduces (iii) to (i).

If E is a module over a local ring A the sets $\mathfrak{m}^k E$, $k \geqslant 0$, form a basis of the neighborhoods of $0 \in E$ for a topology in E. This topology, making E into a topological group, is called the Krull topology of E.

Combining the two previous results we can prove the following

Theorem (Krull). Let A be a local noetherian ring, E a finite module over A. Then:

(i) The Krull topology of E is separated.

(ii) Every submodule F of E is closed in E.

(iii) The topology induced by E in a submodule F is the Krull topology of F.

Proof. (i) Let $F = \bigcap_{k \geq 0} \mathfrak{m}^k E = \overline{\{0\}}$. Then

$$\mathfrak{m}F = \mathfrak{m}\big((\mathfrak{m}^n E) \cap F\big) = (\mathfrak{m}^{n+1} E) \cap F = F$$

by the Artin-Rees lemma. Hence Nakayama's lemma implies that $F = \{ 0 \}$.

(ii) Let $f : E \to E/F$ be the natural map. Then

$$f(\bar{F}) \subset f(F + \mathfrak{m}^k E) = f(\mathfrak{m}^k E) = \mathfrak{m}^k (E/F) .$$

Hence $f(\bar{F}) \subset \cap\, \mathfrak{m}^k (E/F) = \{ 0 \}$, using (i).). But $f(\bar{F}) \subset \{ 0 \}$ is equivalent to $\bar{F} \subset F$.

(iii) It is clear that $\mathfrak{m}^k F \subset (\mathfrak{m}^k E) \cap F$. Hence the Krull topology of F is finer than that induced by E ; in other words the inclusion $F \to E$ is continuous. Conversely the Artin-Rees lemma shows that

$$(\mathfrak{m}^{n+k} E) \cap F = \mathfrak{m}^k \big((\mathfrak{m}^n E) \cap F\big) \subset \mathfrak{m}^k F$$

which proves that the induced topology is finer than the Krull topology of F.

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