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THE PRINCIPLE OF SIGNS

by J. M. BOARDMAN

In algebraic topology one frequently encounters the following algebraic situation. One finds a graded ring, consisting of a sequence of abelian groups A_n ($n=0, 1, 2, \ldots$) and an associative system of bilinear multiplication maps $\mu_{m,n}\colon A_m \times A_n \to A_{m+n}$. It often happens that for geometric reasons this multiplication commutes up to sign,

$$\mu_{m,n}(x, y) = \varepsilon_{m,n}\, \mu_{n,m}(y, x) \qquad (x \in A_m, y \in A_n),$$

where the sign $\varepsilon_{m,n} = \pm 1$ depends only on m and n.

Now the multiplication is usually not canonically defined, to the extent that each $\mu_{m,n}$ may reasonably be changed by a sign depending on m and n. This modification may alter $\varepsilon_{m,n}$ if $m \neq n$, but leaves $\varepsilon_{n,n}$ fixed. This freedom is used to simplify $\varepsilon_{m,n}$ into a memorable form, while retaining associativity. It is not usually possible to abolish the signs completely. In important cases, however, we can arrange $\varepsilon_{m,n} = (-1)^{mn}$. This simple choice is found in practice to work remarkably well, even in much more complicated algebraic structures. It has therefore acquired the status of a metaphysical rule of thumb lacking adequate justification. It is much used, and therefore also much misused, expressed in the form:

Whenever two elements of dimensions m and n are interchanged, one ought to insert the sign $(-1)^{mn}$, in order to make the algebra consistent.

We show that this *Principle of Signs* can be made quite precise.

We take a general algebraic system, consisting of:

(1) *(a)* A class Ω of *elements*.

(b) A class $\mathfrak{A}$ of n-ary *operations* on Ω, for various values $n \geqq 0$, which need not be everywhere defined. (A n-ary operation is a function from some subset of the n-fold cartesian product Ω^n to Ω. A non-empty 0-ary operation is simply an element of Ω.)

(c) An *involution* $\mathrm{T} \in \mathfrak{A}$ (to be thought of as " minus ").

(d) A *dimension* function $|\ |$ defined on the whole of Ω, taking values in the additive cyclic group $\mathbf{Z}_2$ of order 2.

We subject these to the axioms:

(2) *(a)* $TTx = x$ whenever Tx is defined.

(b) $|\alpha(x_1, x_2, \ldots, x_n)| = \sum_{i=1}^{i=n} |x_i|$
whenever $\alpha(x_1, x_2, \ldots, x_n)$ is defined, for all $\alpha \in \mathfrak{A}$.

(c) $\alpha(x_1, x_2, \ldots, Tx_i, \ldots, x_n) = T\alpha(x_1, x_2, \ldots, x_i, \ldots, x_n)$
whenever $\alpha(x_1, x_2, \ldots, x_n)$ is defined, and $1 \leqq i \leqq n$, for all $\alpha \in \mathfrak{A}$.

Given a m-ary operation α, a n-ary operation β, and an integer i $(1 \leqq i \leqq m)$, we construct the composite $(m+n-1)$-ary operation γ by setting

$$\begin{aligned} &\gamma(x_1, x_2, \ldots, x_{i-1}, y_1, y_2, \ldots, y_n, x_{i+1}, \ldots, x_m) \\ &= \alpha(x_1, x_2, \ldots, x_{i-1}, \beta(y_1, y_2, \ldots, y_n), x_{i+1}, \ldots, x_m)\,. \end{aligned} \tag{3}$$

If α and β are in $\mathfrak{A}$, it is clear that γ also satisfies (2). In particular, specialization of the argument x_i in $\alpha(x_1, x_2, \ldots, x_m)$ is included as composition with a 0-ary operation β. It is convenient to enlarge $\mathfrak{A}$ so as to include all operations such as γ obtained by composing operations in $\mathfrak{A}$, and also all those operations obtained by restricting the domain of definition of an operation in $\mathfrak{A}$, subject to (2*c*).

DEFINITION. *We say the identity*

$$\alpha(x_1, x_2, \ldots, x_n) = T^{\varepsilon}\beta(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n}) \tag{4}$$

is in standard form *if* α *and* β *are in* $\mathfrak{A}$, *both sides have the same domain of definition,* π *is a permutation of* $\{1, 2, \ldots, n\}$, *and*

$$\varepsilon = \sum |x_{\pi i}|\,|x_{\pi j}|\,, \tag{5}$$

where we sum over all pairs (i, j) *such that* $i < j$ *and* $\pi i > \pi j$.

Thus the term $|x_{\pi i}|\,|x_{\pi j}|$ appears in (5) if and only if the arguments $x_{\pi i}$ and $x_{\pi j}$ appear in opposite orders on the two sides of (4).

THEOREM. *Suppose we are given an algebraic system as in* (1) *and* (2), *with* $\mathfrak{A}$ *enlarged as indicated above, and a class of identities in standard form. Then all allowable consequent identities are in standard form.*

We allow as consequent identities all those obtained from the given identities in a finite number of steps of any of the following types:

Type 1. We may deduce the trivial identity

$$\alpha(x_1, x_2, \ldots, x_n) = \alpha(x_1, x_2, \ldots, x_n)$$

for any operation $\alpha \in \mathfrak{A}$.

Type 2. *Reversal.* From the identity (4) we deduce the identity

$$\beta(x_1, x_2, \ldots, x_n) = T^{\varepsilon}\alpha(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}),$$

where σ is the inverse permutation to π, by using (2a).

Type 3. Transitivity. From the identities

$$\alpha(x_1, x_2, \ldots, x_n) = T^{\varepsilon}\beta(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n})$$

and

$$\beta(y_1, y_2, \ldots, y_n) = T^{\eta}\gamma(y_{\rho 1}, y_{\rho 2}, \ldots, y_{\rho n})$$

we deduce the identity

$$\alpha(x_1, x_2, \ldots, x_n) = T^{\varepsilon+\eta}\gamma(x_{\pi\rho 1}, x_{\pi\rho 2}, \ldots, x_{\pi\rho n}).$$

Type 4. Substitution. Given a m-ary operation $\omega \in \mathfrak{A}$ and an integer j $(1 \leqq j \leqq m)$, we deduce from (4) the identity

$$\begin{aligned} &\omega(y_1, \ldots, y_{j-1}, \alpha(x_1, x_2, \ldots, x_n), y_{j+1}, \ldots, y_m) \\ &= T^{\varepsilon}\omega(y_1, \ldots, y_{j-1}, \beta(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n}), y_{j+1}, \ldots, y_m) \end{aligned} \tag{6}$$

by using (2c). We use (3) to express this in the form (4).

Type 5. Substitution. Given a m-ary operation $\omega \in \mathfrak{A}$ and an integer i $(1 \leqq i \leqq n)$, we deduce from (4) the identity

$$\begin{aligned} &\alpha(x_1, \ldots, x_{i-1}, \omega(y_1, y_2, \ldots, y_m), x_{i+1}, \ldots, x_n) \\ &= T^{\varepsilon}\beta(x_{\pi 1}, \ldots, \omega(y_1, y_2, \ldots, y_m), \ldots, x_{\pi n}), \end{aligned} \tag{7}$$

by substituting for x_i. We use (3) to express this in the form (4).

Type 6. Cancellation. Suppose the m-ary operation $\omega \in \mathfrak{A}$ satisfies the cancellation law for the argument y_j $(1 \leqq j \leqq m)$, namely that

$$\omega(y_1, y_2, \ldots, y_j, \ldots, y_m) = \omega(y_1, y_2, \ldots, y_j', \ldots, y_m)$$

implies $y_j' = y_j$. Then we deduce the identity (4) from the identity (6), making use of (2c), provided that α and β are in $\mathfrak{A}$ and have the same domain of definition.

Type 7. Synthesis. Given a m-ary operation $\omega \in \mathfrak{A}$ and an integer i $(1 \leqq i \leqq n)$, suppose that whenever $\alpha(x_1, x_2, \ldots, x_n)$ is defined, x_i can be expressed in the form $\omega(y_1, y_2, \ldots, y_m)$. Then we may deduce the identity (4) from the identity (7), provided the operations α and β are in $\mathfrak{A}$ and have the same domain of definition.

Proof of Theorem. We must check that the deduced identities are in standard form whenever the assumed identities are. For types 1, 4 and 6, this is trivial.

It is well known that the signature homomorphism from a permutation group to $\mathbf{Z}_2$ can be given by a formula similar to (5). Our situation is a slight variant. Put $y_i = x_{\pi i}$ in Types 2 and 3. Put $p = \pi j$ and $q = \pi i$ in (5); we find

$$\varepsilon = \sum |x_p|\,|x_q| = \sum |y_{\sigma p}|\,|y_{\sigma q}| \qquad (p < q, \sigma p > \sigma q)\,.$$

This proves that Type 2 yields identities in standard form. Again, this time putting $i = \rho p$ and $j = \rho q$, we find

$$\varepsilon = \sum |x_{\pi i}|\,|x_{\pi j}| = \sum |x_{\pi\rho p}|\,|x_{\pi\rho q}| \qquad (\rho p < \rho q, \pi\rho p > \pi\rho q)$$

and

$$\eta = \sum |y_{\rho i}|\,|y_{\rho j}| = \sum |x_{\pi\rho i}|\,|x_{\pi\rho j}| \qquad (i < j, \rho i > \rho j)\,,$$

summing as indicated. From these we deduce

$$\varepsilon + \eta = \sum |x_{\pi\rho i}|\,|x_{\pi\rho j}| \qquad (i < j, \pi\rho i > \pi\rho j)\,,$$

and hence that Type 3 yields identities in standard form.

As for Types 5 and 7, we see from (2*b*) that we have the required power of T.

This completes the proof.

The standard application is to graded algebras and modules. In this case, Ω is the class of all elements of all the graded groups A, B, ... under consideration, including constructed ones such as $A \otimes B$ and Hom (A, B). The involution T is $-$, and dimension is the grading, reduced modulo 2. The operations usually include $\otimes$, multiplication, module actions, evaluation, composition, etc. Natural transformations are to be regarded as unary operations.

However, the theorem has much scope in situations outside pure algebra, for example in the formal aspects of stable homotopy theory.

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