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1. *Introduction.* For any extended real valued function f defined on $[a, b]$, let $\mathbf{P}_n(f)$ be the class of all polynomials P of degree $\leqq n$ satisfying the condition $P(x) \leqq f(x)$ for all $x \in [a, b]$ and w a non negative Lebesgue integrable function on $[a, b]$ such that $\int_a^b w(t)\,dt > 0$. We say that $P \in \mathbf{P}_n(f)$ is a polynomial of best one sided approximation to f on $[a, b]$ corresponding to the weight function w if

$$\int_a^b w(t) P(t)\,dt = \sup\left\{\int_a^b w(t) Q(t)\,dt : Q \in \mathbf{P}_n(f)\right\} \equiv \Lambda(f)$$

If f is integrable on $[a, b]$, this is equivalent to

$$\int_a^b w(t)(f(t) - P(t))\,dt = \inf\left\{\int_a^b w(t)(f(t) - Q(t))\,dt : Q \in \mathbf{P}_n(f)\right\}.$$

The polynomial P defined here is clearly a polynomial of best approximation to f from below. The polynomial of best approximation to f from above is defined similarly.

The problems of one sided approximation appear frequently in analysis. It is well known (see [1], p. 65, Aufg. 137) that for every Riemann integrable function f on $[a, b]$ and $\varepsilon > 0$ there exist polynomials p and P such that

$$p(x) \leqq f(x) \leqq P(x), \qquad x \in [a, b]$$

and

$$\int_a^b (P(x) - p(x))\,dx < \varepsilon.$$

A special case of this result corresponding to the function

$$f(x) = \begin{cases} 0, 0 \leqq x \leqq e^{-1} \\ x^{-1}, e^{-1} < x \leqq 1 \end{cases}$$

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played a significant role in J. Karamata'spro of [2] of the famous Hardy-Littlewood's Tauberian theorem. By a refinement of Karamata's method, based on a more precise one sided approximation to the same function, G. Freud [3] obtained an estimate of the remainder term in the Hardy-Littlewood's theorem. More general results of this type based on one sided approximation to

$$f(x) = \begin{cases} 0, \; 0 \leqq x \leqq e^{-1} \\ x^{-1}(1+\log x)^{p-1}, \; e^{-1} < x \leqq 1 \end{cases}$$

have been obtained by T. Ganelius [5].

From a more practical point of view, one sided approximation by polynomials to an integrable function f gives immediately upper and lower estimates for the integral of wf.

However, the polynomials of best one sided approximation have not been studied systematically, and they have been found explicitly only in few special cases by G. Freud [4] and T. Ganelius [6].

In the first part of this paper we shall consider the problems of existence and uniqueness of polynomials of best one sided approximation. We shall prove the existence of a polynomial of best one sided approximation to f, of degree $\leqq n$, assuming that f is bounded from below on $[a, b]$ and that f is either Lebesgue integrable on $[a, b]$ or finite on certain subsets of $n+1$ distinct points of $[a, b]$.

While the existence of a polynomial of best one sided approximation has been established under very general hypotheses, it is not difficult to see that a polynomial of best one sided approximation is not necessarily unique even for continuous functions.

Consider for example the function π_n^+ defined by

$$\pi_n^+(x) = \begin{cases} \pi_n(x) \text{ if } \pi_n(x) \geqq 0 \\ 0, \text{ if } \pi_n(x) < 0 \end{cases}$$

where π_n is the orthogonal polynomial of degree n on $[a, b]$ corresponding to the weight function w. It is easy to see by means of the Gauss quadrature formula that for any polynomial Q of degree $\leqq 2n-1$ such that $Q(x) \leqq \pi_n^+(x)$, $x \varepsilon [a, b]$, we have

$$\int_a^b w(t)\, Q(t)\, dt \leqq 0 .$$

On the other hand, we have $\lambda\pi_n(x) \leqq \pi_n^+(x)$ for every $0 \leqq \lambda \leqq 1$ and $x \,\varepsilon\, [a, b]$ and

$$\int_a^b w(t)\, \lambda\pi_n(t)\, dt = 0 .$$

This shows that $\lambda\pi_n$ is a polynomial of best one sided approximation to π_n^+, of degree $\leqq 2n-1$, for every $0 \leqq \lambda \leqq 1$.

Thus, the continuity of a function does not guarantee the uniqueness of its polynomial of best one sided approximation. We shall however prove that a polynomial of best one sided approximation to a differentiable function is necessarily unique.

In the second part of this paper we shall consider the problem of explicitly determining polynomials of best one sided approximation from above and from below.

Our first theorems deal with polynomials of best one sided approximation of degree $\leqq n-1$ to functions whose n-th derivative is of constant sign on (a, b).

We mention in particular polynomials π_* and π^* of best one sided approximation from below and from above, of degree $n-1$, to x^n on $[-1, 1]$, corresponding to the weight function 1:

$$w^*(x) = \begin{cases} x^n - \left(\tilde{P}^{(00)}_{[\frac{1}{2}n]}(x)\right)^2 & \text{if } n \text{ is even} \\ x^n - (1+x)\left(\tilde{P}^{(0,1)}_{[\frac{1}{2}(n-1)]}(x)\right)^2 & \text{if } n \text{ is odd} \end{cases}$$

$$w^*(x) = \begin{cases} x^n - (1-x^2)\left(\tilde{P}^{(1,1)}_{[\frac{1}{2}n]-1}(x)\right)^2 & \text{if } n \text{ is even} \\ x^n - (1-x)\left(\tilde{P}^{(1,0)}_{[\frac{1}{2}(n-1)]}(x)\right)^2 & \text{if } n \text{ is odd} \end{cases}$$

where $\tilde{P}_n^{(\alpha,\beta)}$ is the Jacobi polynomial of degree n, normalized so that the coefficient of x^n is 1. We recall that the Jacobi polynomials $(\tilde{P}_n^{(\alpha,\beta)})$ form an orthogonal sequence on the interval $[-1, 1]$ with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$.

As another application of these results we obtain trigonometric polynomials of best one sided approximation on $[-\pi, \pi]$ to

$$h(\theta) = \sum_{k=1}^{\infty} \lambda_k \cos k\theta$$

where

$$\lambda_k = \int_0^1 t^k \, d\alpha(t) , \qquad k = 1, 2, \ldots$$

with a non decreasing α on [0, 1]. In particular we obtain trigonometric polynomials of best one sided approximation to the even Bernoulli polynomials

$$\tfrac{1}{2}(-1)^m b_{2m}(\theta) = \sum_{k=1}^{\infty} \frac{(2m)!}{k^{2m}} \cos k\theta$$

since

$$\frac{(2m)!}{k^{2m}} = \int_0^1 t^k \, d(-\log^{2m} t) .$$

The trigonometric polynomials of best one sided approximation to Bernoulli polynomials both even and odd have been obtained by T. Ganelius [6] in connection with the problem of one sided approximation to functions whose r-th derivative is of bounded variation.

In addition to results of this type we shall obtain polynomials of best one sided approximation of degree $\leqq n$ to functions of the form $h(x^2)$ on $[-1, 1]$ assuming that the weight function w is even and that the $[\frac{1}{2} n]+1$-th derivative of h is of constant sign on (0,1). Choosing in particular $h(t) = \sqrt{t}$ we obtain polynomials of best one sided approximation from above and from below to $|x|$, of any degree. The polynomial of best one sided approximation to $|x|$ from below of degree $\leqq 4n+1$ has been obtained already by G. Freud [4] by a different procedure based on certain results of T. J. Stieltjes and A. A. Markov.

2. *Existence and uniqueness of polynomials of best one sided approximation.* (i) The proofs of the existence theorems are based on the following well known results:

LEMMA 1. *If (Q_m) is a sequence of polynomials of degree $\leqq n$ such that*

$$\int_a^b w(t) \, | Q_m(t) | \, dt \leqq M, \; m = 1, 2, \ldots$$

then there exists a subsequence (Q_{m_k}) converging to a polynomial Q of degree $\leqq n$ and the convergence is uniform on every finite interval.

LEMMA 2. *If* (Q_m) *is a sequence of polynomials of degree* $\leqq n$*, which is bounded at* n+1 *distinct points, then there exists a subsequence* (Q_{m_k}) *converging to a polynomial* Q *of degree* $\leqq n$ *and the convergence is uniform on every finite interval.*

Our first result can be stated as follows:

THEOREM 1. *If* f *is bounded from below and Lebesgue integrable on* $[a, b]$*, then the polynomial of best one sided approximation to* f *on* $[a, b]$*, of degree* $\leqq n$*, exists.*

Proof. Since f is bounded from below and

$$Q \varepsilon \mathbf{P}_n(f) \Rightarrow \int_a^b w(t)\,Q(t)\,dt \leqq \int_a^b w(t) f(t)\,dt < \infty ,$$

we have $-\infty < \Lambda(f) < \infty$.

Let (Q_m) be a sequence of polynomials in $\mathbf{P}_n(f)$ such that

$$\Lambda_m = \int_a^b w(t)\,Q_m(t)\,dt \rightarrow \Lambda(f) \quad (m \rightarrow \infty) .$$

We have then

$$\begin{aligned}\int_a^b w(t)\,|\,Q_m(t)\,|\,dt &\leqq \int_a^b w(t)(f(t) - Q_m(t))\,dt + \int_a^b w(t)\,|f(t)\,|\,dt \\ &\leqq 2\int_a^b w(t)\,|f(t)\,|\,dt - \Lambda_m \\ &\leqq M\end{aligned}$$

for all $m = 1, 2, \ldots$

By Lemma 1, the sequence (Q_m) contains a uniformly convergent subsequence (Q_{m_k}) on $[a, b]$ converging to a polynomial P of degree $\leqq n$.

Since $Q_{m_k}(x) \leqq f(x)$, it follows that $P(x) \leqq f(x)$ for all $x \in [a, b]$ and so $P \in \mathbf{P}_n(f)$. On the other hand, from

$$\int_a^b w(t)\,P(t)\,dt = \lim_{k \rightarrow \infty} \int_a^b w(t)\,Q_{m_k}(t)\,dt = \Lambda(f)$$

follows that P is the polynomial of best one sided approximation to f on $[a, b]$.

The following existence theorem requires only that f be finite on certain subsets of $n+1$ points of $[a, b]$.

THEOREM 2. *Let* $\{\xi_0, \ldots, \xi_n\}$ *be* $n+1$ *distinct points of* $[a, b]$ *such that for any polynomial* R *of degree* $\leqq n$ *we have*

$$\int_a^b w(t)\, R(t)\, dt = \sum_{\nu=0}^{n} W_\nu^n R(\xi_\nu)$$

with $W_\nu^n > 0$, $\nu = 0, 1, \ldots, n$.

Let f *be an extended real valued function on* $[a, b]$ *which is bounded from below on* $[a, b]$ *and finite at the points* $\{\xi_0, \ldots, \xi_n\}$.

Then the polynomial of best one sided approximation to f *on* $[a, b]$ *of degree* $\leqq n$ *exists.*

Proof. Since f is finite on $\{\xi_0, \ldots, \xi_n\}$ we can find $M>0$ such that

$$f(\xi_\nu) \leqq M\,, \qquad \nu = 0, \ldots, n\,. \tag{2.1}$$

Since f is bounded from below we have clearly $\Lambda(f) > -\infty$. On the other hand, for any $Q \in \mathbf{P}_n(f)$ we have by (2.1)

$$\int_a^b w(t)\, Q(t)\, dt = \sum_{\nu=0}^{n} W_\nu^n Q(\xi_\nu) \leqq \sum_{\nu=0}^{n} W_\nu^n f(\xi_\nu) \leqq M \int_a^b w(t)\, dt\,.$$

Thus $-\infty < \Lambda(f) < \infty$.

Next, let (Q_m) be a sequence of polynomials in $\mathbf{P}_n(f)$ such that

$$\Lambda_m = \int_a^b w(t)\, Q_m(t)\, dt \to \Lambda(f) \qquad (m \to \infty)\,.$$

Since

$$Q_m(\xi_\nu) \leqq f(\xi_\nu) \leqq M, \quad \nu = 0, \ldots, n$$

all sequences $(Q_m(\xi_\nu))$, $\nu = 0, \ldots, n$ are bounded from above. We shall prove that they are also bounded from below.

Assume that at least one of these sequences, $(Q_m(\zeta_r))$, $0 \leqq r \leqq n$, is not bounded from below. Then there exists a subsequence $(Q_{m_k}(\xi_r))$ such that

$$Q_{m_k}(\xi_r) \to -\infty \quad (k \to \infty)\,.$$

This implies that

$$\Lambda_{m_k} = \int_a^b w(t)\, Q_{m_k}(t)\, dt = \sum_{\nu=0}^{n} W_\nu^n Q_{m_k}(\xi_\nu)$$

$$= W_r^n Q_{m_k}(\xi_r) + \sum_{\substack{\nu=0 \\ (\nu \neq r)}}^{n} W_\nu^n Q_{m_k}(\xi_\nu)$$

$$\leqq W_r^n Q_{m_k}(\xi_r) + M \int_a^b w(t)\, dt$$

and since $W_r^n > 0$, it would follow that $\Lambda_{m_k} \to -\infty$ $(k \to \infty)$ which is impossible.

Thus, there exists a constant $K > 0$ such that

$$| Q_m(\xi_\nu) | \leqq K, \quad \nu = 0, \ldots, n, \quad m = 1, 2, \ldots$$

The rest of the proof, based on Lemma 2, is the same as in Theorem 1.

(ii) In order to simplify the proof of the uniqueness theorem we shall introduce the concept of the point of contact.

Let f be continuous on $[a, b]$ and let Q be a polynomial such that $Q(x) \leqq f(x)$ for all $x \in [a, b]$. Any point $x_0 \in [a,b]$ such that $Q(x_0) = f(x_0)$ will be called a point of contact.

The proof of the uniqueness theorem is based on two lemmas. The first lemma states that the number of points of contact of the polynomial of best one sided approximation to f on $[a, b]$, of degree $\leqq n$, is $\geqq [\frac{1}{2} n] + 1$. The second lemma states that none or at most one of the end points of the interval $[a, b]$ can be a point of contact, according to wether n is odd or even, if the number of points of contact is exactly equal to $[\frac{1}{2} n] + 1$.

LEMMA 3. *If f is continuous on $[a, b]$ and P a polynomial of best one sided approximation to f on $[a, b]$ of degree $\leqq n$, then there exist at least $[\frac{1}{2} n] + 1$ distinct points of contact in $[a, b]$.*

Proof. The lemma is obviously true if $n = 0$ or $n = 1$. Thus we can assume that $n \geqq 2$.

We have $f(x) - P(x) \geqq 0$ for all $x \in [a, b]$ and the equality sign holds clearly for at least one $x \in [a, b]$. Assume that there are at most $k < [\frac{1}{2} n] + 1$ points of contact $x_1 < \ldots < x_k$ in $[a, b]$.

Let $\varepsilon>0$ be such that $2\varepsilon < \min_{1 \leqq i \leqq k-1} (x_{i+1}-x_i)$ and let Q_ε be the polynomial

$$Q_\varepsilon(x) = (x-(x_1-\varepsilon))(x-(x_1+\varepsilon)) \ldots (x-(x_k-\varepsilon))(x-(x_k+\varepsilon)).$$

Since $k < [\frac{1}{2} n] + 1$ we have $k \leqq [\frac{1}{2} n]$ and so

$$\deg Q_\varepsilon = 2k \leqq 2[\tfrac{1}{2} n] \leqq n .$$

Since

$$\lim_{\varepsilon \to 0} Q_\varepsilon(x) = (x-x_1)^2 \ldots (x-x_k)^2$$

uniformly on $[a, b]$ and

$$\int_a^b w(x)(x-x_1)^2 \ldots (x-x_k)^2 \, dx > 0$$

we can choose $\varepsilon>0$ such that

$$\int_a^b w(t) Q_\varepsilon(t) \, dt > 0 . \tag{2.2}$$

Let $\mathscr{I} = \bigcup_{\nu=1}^{n} (x_\nu-\varepsilon, x_\nu+\varepsilon)$. For $x \in [a, b]\setminus\mathscr{I}$ we have $f(x)-P(x) > 0$ and since both functions are continuous on the compact set $[a,b]\setminus\mathscr{I}$ we can find a $d>0$ such that

$$f(x) - P(x) \geqq d$$

for all $x \in [a, b]\setminus\mathscr{I}$.

Let

$$\eta = d \,/ \max_{a \leqq x \leqq b} |Q_\varepsilon(x)| .$$

We shall show that the polynomial $P+\eta Q_\varepsilon$ is in $\mathbf{P}_n(f)$ and that it approximates f from below on $[a, b]$ better than P.

We have clearly $\deg(P+\eta Q_\varepsilon) \leqq n$. On $[a, b] \cap \mathscr{I}$ we have $Q_\varepsilon(x) \leqq 0$ and consequently

$$\eta Q_\varepsilon(x) \leqq 0 \leqq f(x)-P(x) .$$

On $[a, b]\setminus\mathscr{I}$ we have

$$\eta Q_\varepsilon(x) \leqq d \leqq f(x)-P(x) .$$

Thus $P + \eta Q_\varepsilon$ is in $\mathbf{P}_n(f)$. On the other hand, from (2.2) follows that

$$\int_a^b w(t)\,P(t)\,dt < \int_a^b w(t)\big(P(t)+\eta Q_\varepsilon(t)\big)\,dt\,.$$

This however contradicts the hypothesis that P is a polynomial of best one sided approximation to f on $[a, b]$ of degree $\leqq n$. Consequently, we have $k \geqq [\frac{1}{2}n]+1$ and the lemma is proved.

LEMMA 4. *Suppose that f is continuous on $[a, b]$ and that P is a polynomial of best one sided approximation to f on $[a, b]$ of degree $\leqq n$. Suppose also that there are exactly $[\frac{1}{2}n] + 1$ points of contact on $[a, b]$.*

If n is odd, then all $[\frac{1}{2}\mathrm{n}] + 1$ points of contact are in the interior of $[a, b]$. On the other hand, if n is even, then at least $[\frac{1}{2}n]$ points of contact are in the interior of $[a, b]$.

Proof. Assume first that n is odd, $n = 2l-1$ $(l\geqq 2)$. We have then $[\frac{1}{2}n]+1 = l$ points of contact $a \leqq x_1 < \ldots < x_l \leqq b$. Suppose that $x_1 = a$. For each $\varepsilon>0$ such that $2\varepsilon< \min\limits_{1\leqq i\leqq l-1}(x_{i+1}-x_i)$ we define Q_ε by

$$Q_\varepsilon(x) = \big(x-(a+\varepsilon)\big)\big(x-(x_2-\varepsilon)\big)\big(x-(x_2+\varepsilon)\big)\ldots\big(x-(x_l-\varepsilon)\big)\big(x-(x_l+\varepsilon)\big)\,.$$

We have clearly deg $Q_\varepsilon = 2l-1 = n$. Since

$$\lim_{\varepsilon\to 0} Q_\varepsilon(x) = (x-a)(x-x_2)^2\ldots(x-x_l)^2$$

uniformly on $[a, b]$ and

$$\int_a^b w(x)(x-a)(x-x_2)^2\ldots(x-x_l)^2\,dx > 0\,.$$

we can choose $\varepsilon>0$ such that

$$\int_a^b w(t)\,Q_\varepsilon(t)\,dt > 0\,.$$

As in the proof of Lemma 3, we can find $\eta>0$ so small that

$$\eta Q_\varepsilon(x) \leqq f(x)-P(x)$$

for all $x \in [a, b]$. This shows that $P+\eta Q_\varepsilon$ is in $\mathbf{P}_n(f)$. On the other hand, we have

$$\int_a^b w(t)\,P(t)\,dt < \int_a^b w(t)\big(P(t)+\eta Q_\varepsilon(t)\big)\,dt$$

which is impossible.

If $l = 1$, the same argument can be used if we define Q_ε by $Q_\varepsilon(x) = x-(a+\varepsilon)$.

Using a similar argument we find that b cannot be a point of contact.

Next, assume that n is even, $n = 2l$, $l \geqq 2$ and that both a and b are points of contact. We have then $[\frac{1}{2}n]+1 = l+1$ points of contact $a = x_1<x_2<...<x_l<x_{l+1} = b$. For each $\varepsilon>0$ such that $2\varepsilon < \min\limits_{1\leqq i\leqq l}(x_{i+1}-x_i)$ we define Q_ε by

$$Q_\varepsilon(x) = \big(x-(a+\varepsilon)\big)\big(x-(x_2-\varepsilon)\big)\big(x-(x_2+\varepsilon)\big)\ldots$$
$$\ldots\big(x-(x_l-\varepsilon)\big)\big(x-(x_l+\varepsilon)\big)(b-\varepsilon-x)\,.$$

We have

$$\deg Q_\varepsilon = 2l = n\,.$$

Since

$$\lim_{\varepsilon\to 0} Q_\varepsilon(x) = (x-a)(x-x_2)^2\ldots(x-x_l)^2(b-x)$$

uniformly on $[a, b]$ and

$$\int_a^b w(x)(x-a)(x-x_2)^2\ldots(x-x_l)^2(b-x)\,dx > 0$$

we can choose $\varepsilon>0$ such that

$$\int_a^b w(t)\,Q_\varepsilon(t)\,dt > 0\,.$$

As in the proof of Lemma 3, we can choose $\eta>0$ so small that

$$\eta Q_\varepsilon(x) \leqq f(x) - P(x)$$

for all $x \in [a, b]$. This means that $P+\eta Q_\varepsilon$ is in $\mathbf{P}_n(f)$ and as it is easy to see that $P+\eta Q_\varepsilon$ approximates f from below on $[a, b]$ better than P, which is impossible. Thus, a and b cannot be both points of contact and consequently we have in this case at least $[\frac{1}{2}n]$ points of contact in the interior of $[a, b]$.

If $l=1$, the same argument can be used if we define Q_ε by

$$Q_\varepsilon(x) = \big(x-(a+\varepsilon)\big)(b-\varepsilon-x)\,.$$

The uniqueness theorem can be stated as follows.

THEOREM 3. *If f is continuous on $[a, b]$ and differentiable on (a, b), then the polynomial of best one sided approximation to f is unique.*

Proof. Suppose that P_1 and P_2 are two polynomials of best one sided approximation to f in $\mathbf{P}_n(f)$. Then $\frac{1}{2}(P_1+P_2)$ is also a polynomial of best one sided approximation to f in $\mathbf{P}_n(f)$.

Thus, by Lemma 3, there are $k \geqq [\frac{1}{2}n] + 1$ distinct points $x_1, ..., x_k$ in $[a, b]$ such that

$$(2.3) \qquad f(x_i) = \tfrac{1}{2}(P_1(x_i) + P_2(x_i)), \qquad i = 1, ..., k .$$

Since $P_j(x_i) \leqq f(x_i)$ $i = 1, ..., k$, $j = 1, 2$, from (2.3) follows that

$$(2.4) \qquad P_1(x_i) = f(x_i) = P_2(x_i), \qquad i = 1, ..., k .$$

Since $f(x) - P_j(x) \geqq 0$, $x \in [a, b]$, the function $f - P_j$ in view of (2.4) assumes its minimum value at $x_1, ..., x_k$.

Suppose first that $k > [\frac{1}{2}n] + 1$, i.e. $k \geqq [\frac{1}{2}n] + 2$. Then we have at least $[\frac{1}{2}n]$ points of contact $x_2, ..., x_{k-1}$ in the interior of $[a, b]$. Since $f - P_j$ assumes its minimum value at these points and $f' - P_j'$ exists in (a, b), it follows that

$$(2.5) \qquad P_1'(x_i) = f'(x_i) = P_2'(x_i), \qquad i = 2, ..., k-1 .$$

Since

$$2k-2 \geqq 2([\tfrac{1}{2}n]+2) - 2 = 2([\tfrac{1}{2}n]+1) \geqq n+1$$

and since both P_1 and P_2 are polynomials of degree $\leqq n$, from (2.4) and (2.5) follows that $P_1 = P_2$.

Next, suppose that $k = [\frac{1}{2}n]+1$ and that n is odd. By Lemma 4 all $[\frac{1}{2}n]+1$ points of contact are in the interior of $[a, b]$. Consequently, we have

$$(2.6) \qquad P_1'(x_i) = f'(x_i) = P_2'(x_i), \qquad i = 1, ..., k .$$

In this case we have $2k = 2([\frac{1}{2}n]+1) = n+1$ conditions and from (2.4) and (2.6) it follows that $P_1 = P_2$.

Finally, suppose that $k = [\frac{1}{2}n]+1$ and that n is even. By Lemma 4, we have at least $[\frac{1}{2}n]$ points of contact in the interior of $[a, b]$ and P_1' and P_2' have the same values at these points. From this fact and (2.4) follows that we have at least $2[\frac{1}{2}n]+1 = n+1$ conditions and again we conclude that $P_1 = P_2$.

3. *Polynomials of best one sided approximation to differentiable functions.*

(i) In this section we shall consider first functions whose n-th derivative is of constant sign on (a, b) and we shall determine explicitly their polynomials of best one sided approximation of degree $\leqq n-1$ corresponding to the weight function w.

Our proofs are based essentially on a remainder formula for Hermite's interpolation and on certain quadrature formulas of highest possible degree of precision.

Remainder Theorem [7]. Let $x_1 < \ldots < x_n$ be n points in $[a, b]$, $m_1, \ldots, m_n$ non negative integers, $N = m_1 + \ldots + m_n + n$ and H_f the polynomial of degree $\leqq N-1$ defined by

$$H_f^{(k)}(x_i) = f^{(k)}(x_i), \qquad k = 0,1, \ldots, m_i, \qquad i = 1, \ldots, n$$

where f is continuous on $[a, b]$ and $f^{(N)}$ exists on (a, b). Then for every $x \in [a, b]$ there exists a ξ such that $\min(x, x_1) < \xi < \max(x, x_n)$ and

$$f(x) - H_f(x) = \frac{f^{(N)}(\xi)}{N!}(x - x_1)^{m_1+1} \ldots (x - x_n)^{m_n+1}.$$

Quadrature formulas [8]. Let (π_m) be the sequence of orthogonal polynomials on $[a, b]$ corresponding to the weight function w.

We shall denote by $(\pi_m^{(\alpha,\beta)})$ the sequence of polynomials which are orthogonal on $[a, b]$ with respect to the weight function

$$(b-x)^\alpha (x-a)^\beta w(x)$$

$(\alpha > -1, \beta > -1)$. Actually, we need only the polynomials $(\pi_m^{(0,0)})$, $(\pi_m^{(0,1)})$, $(\pi_m^{(1,0)})$ and $(\pi_{m-1}^{(1,1)})$; these polynomials can be expressed easily in terms of (π_m). We have $\pi_m^{(0,0)} = \pi_m$,

$$\pi_m^{(0,1)}(x) = \frac{1}{x-a}\begin{vmatrix} \pi_{m+1}(x) & \pi_{m+1}(a) \\ \pi_m(x) & \pi_m(a) \end{vmatrix}, \quad \pi_m^{(1,0)}(x) = \frac{1}{x-b}\begin{vmatrix} \pi_{m+1}(x) & \pi_{m+1}(b) \\ \pi_m(x) & \pi_m(b) \end{vmatrix}$$

and

$$\pi_{m-1}^{(1,1)}(x) = \frac{1}{(x-a)(x-b)}\begin{vmatrix} \pi_{m+1}(x) & \pi_{m+1}(a) & \pi_{m+1}(b) \\ \pi_m(x) & \pi_m(a) & \pi_m(b) \\ \pi_{m-1}(x) & \pi_{m-1}(a) & \pi_{m-1}(b) \end{vmatrix}.$$

If $a = -1$, $b = 1$ and $w(x) = 1$, $x \in [-1, 1]$, the polynomial $\pi_m^{(\alpha,\beta)}$ is equal to the Jacobi polynomial $P_m^{(\alpha,\beta)}$ except for a constant factor. We have in particular

$$P_m^{(0,0)}(x) = P_m(x)$$

$$P_m^{(0,1)}(x) = \frac{P_{m+1}(x) + P_m(x)}{x+1}$$

$$P_m^{(1,0)}(x) = \frac{P_{m+1}(x) - P_m(x)}{x-1}$$

$$P_{m-1}^{(1,1)}(x) = P'_m(x)$$

where P_m is the Legendre polynomial of degree m. On the other hand, if $w(x) = (1-x^2)^{-\frac{1}{2}}$, $x \in (-1, 1)$, the polynomial $\pi_m^{(\alpha,\beta)}$ is equal to the Jacobi polynomial $P_m^{(\alpha-\frac{1}{2},\, \beta-\frac{1}{2})}$ except for a constant factor. We have in particular

$$P_m^{(-\frac{1}{2},-\frac{1}{2})}(x) = c_m T_m(x) = c_m \cos m\theta$$

$$P_m^{(-\frac{1}{2},\frac{1}{2})}(x) = c_m \frac{\cos(m+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}$$

$$P_m^{(\frac{1}{2},-\frac{1}{2})}(x) = c_m \frac{\sin(m+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$$

$$P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(x) = 2c_m U_{m-1}(x) = 2c_m \frac{\sin m\theta}{\sin\theta}$$

where $x = \cos\theta$ and $c_m = \dfrac{1 \cdot 3 \ldots (2m-1)}{2 \cdot 4 \ldots (2m)}$ (see [9], p. 60).

The zeros of the polynomials $\pi_m^{(0,0)}$, $\pi_{m-1}^{(1,1)}$, $\pi_m^{(0,1)}$ and $\pi_m^{(1,0)}$ play an important role in the construction of quadrature formulas of highest possible degree of precision.

We have first the well known Gauss quadrature formula:

I. Let $x_1, \ldots, x_m$ be the zeros of $\pi_m^{(0,0)}$. Then for every polynomial Q of degree $\leq 2m-1$ we have

$$\int_a^b w(t)\, Q(t)\, dt = \sum_{\nu=1}^{m} A_\nu^m Q(x_\nu) \tag{3.1}$$

where $A_\nu^m > 0$, $\nu = 1, \ldots, m$.

In addition to this we shall need the following quadrature formulas obtained independently by A. Markov [10] and R. Radau [11]; the first of these formulas is also attributed to R. Lobatto [12].

II. Let $y_1, ..., y_{m-1}$ be the zeros of $\pi_{m-1}^{(1,1)}$. Then for any polynomial Q of degree $\leqq 2m-1$ we have

(3.2) $$\int_a^b w(t)\,Q(t)\,dt = B_o^m Q(a) + B_m^m Q(b) + \sum_{\nu=1}^{m-1} B_\nu^m Q(y_\nu)$$

where $B_\nu^m > 0$, $\nu = 0, ..., m$.

III. Let $\xi_1, ..., \xi_m$ be the zeros of $\pi_m^{(0,1)}$. Then for any polynomial Q of degree $\leqq 2m$ we have

(3.3) $$\int_a^b w(t)\,Q(t)\,dt = C_o^m Q(a) + \sum_{\nu=1}^{m} C_\nu^m Q(\xi_\nu)$$

where $C_\nu^m > 0$, $\nu = 0, ..., m$.

IV. Let $\eta_1, ..., \eta_m$ be the zeros of $\pi_m^{(1,0)}$. Then for any polynomial Q of degree $\leqq 2m$ we have

(3.4) $$\int_a^b w(t)\,Q(t)\,dt = D_o^m Q(b) + \sum_{\nu=1}^{m} D_\nu^m Q(\eta_\nu)$$

where $D_\nu^m > 0$, $\nu = 0, ..., m$.

We shall consider first polynomials of best one sided approximation to a function whose n-th derivative is non negative.

THEOREM 4. *Assume that f is continuous on $[a, b]$ and that $f^{(n)}(x) \geqq 0$ for all $x \in (a, b)$.*

A. The polynomial P of best approximation to f from below on $[a, b]$ of degree $\leqq n-1$ is defined as follows:

If $n = 2l$, then

(3.5) $$P(x_\nu) = f(x_\nu), \quad P'(x_\nu) = f'(x_\nu), \quad \nu = 1, ..., l$$

where $x_1, ..., x_l$ are the zeros of $\pi_l^{(0,0)}$.

If $n = 2l+1$, then

(3.6) $P(a) = f(a), \quad P(\xi_\nu) = f(\xi_\nu), \quad P'(\xi_\nu) = f'(\xi_\nu), \quad \nu = 1, \ldots, l$

where $\xi_1, \ldots, \xi_l$ *are the zeros of* $\pi_l^{(0,1)}$.

B. The polynomial P *of best approximation to* f *from above on* [a, b] *of degree* $\leqq$ n−1 *is defined as follows:*

If n = 2*l*, *then*

(3.7) $$P(a) = f(a), \quad P(b) = f(b), \quad P(y_\nu) = f(y_\nu), \quad P'(y_\nu) = f(y_\nu), \quad \upsilon = 1, \ldots, l-1$$

where $y_1, \ldots, y_{l-1}$ *are the zeros of* $\pi_{l-1}^{(1,1)}$.

If $n = 2l+1$, *then*

(3.8) $P(b) = f(b), \quad P(\eta_\nu) = f(\eta_\nu), \quad P'(\eta_\nu) = f'(\eta_\nu), \quad \nu = 1, \ldots, l$

where $\eta_1, \ldots, \eta_l$ *are the zeros of* $\pi_l^{(1,0)}$.

Remark. If $f^{(n)}(x) \leqq 0$ for all $x \in (a, b)$, then (3.5) and (3.6) define the polynomial of best approximation to f from *above*, while (3.7) and (3.8) define the polynomial of best approximation to f from *below*.

Proof. A. Assume first that $n = 2l$. If P is the polynomial defined by (3.5) we have clearly deg $P \leqq 2l-1 = n-1$ and by the remainder theorem

$$f(x) - P(x) = \frac{f^{(2l)}(\xi)}{(2l)!}(x-x_1)^2 \ldots (x-x_l)^2 \geqq 0$$

i.e. $P(x) \leqq f(x)$, $x \in [a, b]$. Further, from (3.5) and (3.1) follows that

$$\int_a^b w(t) P(t)\, dt = \sum_{\nu=1}^{l} A_\nu^l f(x_\nu).$$

On the other hand, for any polynomial Q of degree $\leqq 2l-1 = n-1$ such that $Q(x) \leqq f(x)$, $x \in [a, b]$ we have

$$\int_a^b w(t) Q(t)\, dt \leqq \sum_{\nu=1}^{l} A_\nu^l f(x_\nu).$$

If $n = 2l+1$ and P is defined by (3.6), we have $\deg P \leqq 2l = n-1$ and

$$f(x) - P(x) = \frac{f^{(2l+1)}(\xi)}{(2l+1)!}(x-a)(x-\xi_1)^2 \dots (x-\xi_l)^2 \geqq 0$$

i.e. $P(x) \leqq f(x)$, $x \in [a, b]$. The rest of the proof is the same as before, except that we use the quadrature formula (3.3) instead of (3.1).

B. First assume that $n = 2l$. The polynomial P defined by (3.7) is of degree $\leqq 2l-1 = n-1$ and

$$f(x) - P(x) = \frac{f^{(2l)}(\xi)}{(2l)!}(x-a)(x-b)(x-y_1)^2 \dots (x-y_{l-1})^2 \leqq 0$$

.e. $P(x) \geqq f(x)$, $x \in [a, b]$. From (3.7) and (3.2) follows that

$$\int_a^b w(t) P(t)\, dt = B_o^l f(a) + B_l^l f(b) + \sum_{\nu=1}^{l-1} B_\nu^l f(y_\nu) .$$

On the other hand, for any polynomial Q of degree $2(l-1)+1 = 2l-1 = n-1$ such that $Q(x) \geqq f(x)$, $x \in [a, b]$ we have

$$\int_a^b w(t) Q(t)\, dt \geqq B_o^l f(a) + B_l^l f(b) + \sum_{\nu=1}^{l-1} B_\nu^l f(y_\nu) .$$

If $n = 2l+1$ and P is defined by (3.8), we have $\deg P \leqq 2l = n-1$ and

$$f(x) - P(x) = \frac{f^{(2l+1)}(\xi)}{(2l+1)!}(x-b)(x-\eta_1)^2 \dots (x-\eta_l)^2 \leqq 0$$

i.e. $f(x) \geqq P(x)$, $x \in [a, b]$. The remaining part of the proof is the same as before, except that we use the quadrature formula (3.4) instead of (3.2).

(ii) From the Theorem 4 we obtain immediately the polynomials P_* and P^* of best approximation from below and from above to x^n on $[a, b]$, of degree $\leqq n-1$, corresponding to the weight function w:

$$P_*(x) = \begin{cases} x^n - (\tilde{\pi}_l^{(0,0)}(x))^2 & \text{if} \quad n = 2l \\ x_n - (x-a)(\tilde{\pi}_l^{(0,1)}(x))^2 & \text{if} \quad n = 2l+1 \end{cases}$$

and

$$P^*(x) = \begin{cases} x^n + (x-a)(b-x)\left(\tilde{\pi}_{l-1}^{(1,1)}(x)\right)^2 & \text{if} \quad n = 2l \\ x^n + (b-x)\left(\tilde{\pi}_l^{(1,0)}(x)\right)^2 & \text{if} \quad n = 2l+1 \end{cases}$$

Here $\tilde{\pi}_l^{(\alpha,\beta)}$ is the polynomial $\pi_l^{(\alpha,\beta)}$ normalized so that the coefficient of x^n is 1.

As another application of the Theorem 4 we shall obtain certain results about trigonometric polynomials of best one sided approximation.

Let h be a real valued function defined and bounded from below on $[a, b] \subseteq [-\pi,\pi]$. We shall denote by $\mathbf{T}_n(h)$ the class of all trigonometric polynomials

$$q(x) = \tfrac{1}{2} a_0 + \sum_{v=1}^{n} (a_v \cos vx + b_v \sin vx)$$

of degree $\leqq n$ such that $q(x) \leqq h(x)$ for all $x \in [a, b]$.

A trigonometric polynomial $p \in \mathbf{T}_n(h)$ is the polynomial of best approximation from below to h on $[a, b]$ if

$$\int_a^b p(t)\, dt = \sup_{q \in \mathbf{T}_n(h)} \int_a^b q(t)\, dt .$$

The trigonometric polynomial of best approximation from above is defined similarly.

We have the following result:

THEOREM 5. Let (λ_n) *be a sequence of real numbers such that*

$$(3.9) \qquad \lambda_n = \int_0^1 t^n\, d\alpha(t), \qquad n = 1, 2, \ldots$$

with a non decreasing α *on* $[0, 1]$ *and*

$$(3.10) \qquad \sum_{k=1}^{\infty} \lambda_k < \infty .$$

Let h *be defined by*

$$h(\theta) = \sum_{k=1}^{\infty} \lambda_k \cos k\theta .$$

A. The trigonometric polynomial of best approximation to h from below on $[0, \pi]$ *of degree* $\leq n-1$ *is the cosine polynomial p defined as follows:*

If $n = 2l$, *then*

$$p\left(\frac{2k-1}{2l}\pi\right) = h\left(\frac{2k-1}{2l}\pi\right), \quad p'\left(\frac{2k-1}{2l}\pi\right) = h'\left(\frac{2k-1}{2l}\pi\right), \tag{3.11}$$

$$k = 1, \ldots, l$$

If $n = 2l+1$, *then*

$$p(\pi) = h(\pi), \quad p\left(\frac{2k-1}{2l+1}\pi\right) = h\left(\frac{2k-1}{2l+1}\pi\right), \tag{3.12}$$

$$p'\left(\frac{2k-1}{2l+1}\pi\right) = h'\left(\frac{2k-1}{2l+1}\pi\right), \quad k = 1, \ldots, l.$$

B. The trigonometric polynomial of best approximation to h from above on $[0, \pi]$ *of degree* $\leq n-1$ *is the cosine polynomial p defined as follows:*

If $n = 2l$, *then*

$$p(0) = h(0), \quad p(\pi) = h(\pi), \tag{3.13}$$

$$p\left(\frac{2k}{2l}\pi\right) = h\left(\frac{2k}{2l}\pi\right), \quad p'\left(\frac{2k}{2l}\pi\right) = h'\left(\frac{2k}{2l}\pi\right), \quad k = 1, \ldots, l-1.$$

If $n = 2l+1$, *then*

$$p(0) = h(0), \quad p\left(\frac{2k}{2l+1}\pi\right) = h\left(\frac{2k}{2l+1}\pi\right), \tag{3.14}$$

$$p'\left(\frac{2k}{2l+1}\pi\right) = h'\left(\frac{2k}{2l+1}\pi\right), \quad k = 1, \ldots, l.$$

Remark. Since both p and h are even functions, the polynomial p is the trigonometric polynomial of best one sided approximation to h also on the interval $[-\pi, \pi]$.

Proof. For $|x| \leq 1$ let f be defined by

$$f(x) = h(\arccos x).$$

We have then

$$f(x) = \sum_{k=1}^{\infty} \lambda_k T_k(x), \qquad \text{for all } x \in [-1, 1]$$

where (T_n) is the sequence of Tchebishev polynomials. The function f is clearly continuous on $[-1, 1]$.

From the well known expansion

$$\frac{tx - t^2}{1 - 2tx + t^2} = \sum_{k=1}^{\infty} t^k T_k(x)$$

which is valid for $|x| < 1$, $|t| < 1$, and the hypotheses (3.9) and (3.10) follows that for $|x| < 1$

$$\int_0^1 \frac{tx - t^2}{1 - 2tx + t^2} d\alpha(t) = \sum_{k=1}^{\infty} \lambda_k T_k(x) = f(x).$$

Differentiating we get

$$f^{(n)}(x) = n!\, 2^{n-1} \int_0^1 \frac{t^n(1-t)}{(1-2tx+t^2)^{n+1}} d\alpha(t) \geqq 0.$$

We shall first construct the trigonometric polynomial of best approximation to h from below.

A. By Theorem 4, A, the polynomial P of best approximation to f from below on $[-1, 1]$, of degree $\leqq n-1$, corresponding to the weight function $w(x) = (1-x^2)^{-\frac{1}{2}}$, is defined as follows:

If $n = 2l$, then

$$P(x_k) = f(x_k), \qquad P'(x_k) = f'(x_k), \qquad k = 1, \dots, l$$

where $x_1, \dots, x_l$ are the zeros of the Tchebishev polynomial T_l i.e. $P_l^{(-\frac{1}{2}, -\frac{1}{2})}$:

$$x_k = \cos\left(\frac{2k-1}{2l}\pi\right), \qquad k = 1, \dots, l.$$

If $n = 2l+1$, then

$$P(-1) = f(-1), \qquad P(\xi_k) = f(\xi_k), \qquad P'(\xi_k) = f'(\xi_k), \qquad k = 1, \dots, l$$

where $\xi_1, \dots, \xi_l$ are the zeros of the Jacobi polynomial $P_l^{(-\frac{1}{2}, \frac{1}{2})}$:

$$\xi_k = \cos\left(\frac{2k-1}{2l+1}\pi\right), \qquad k = 1, \dots, l.$$

Let $p(\theta) = P(\cos\theta)$. Then p is clearly a cosine polynomial of degree $\leqq n-1$. From

$$P(x) \leqq h(\arccos x), \qquad x \in [-1, 1]$$

follows that $p(\theta) \leqq h(\theta)$, $\theta \in [0, \pi]$ and it is easy to see that the polynomial p satisfies conditions (3.11) or (3.12) according to whether n is even or odd.

Finally

$$\int_0^{\pi} p(\theta)\, d\theta = \int_{-1}^{+1} (1-x^2)^{-\frac{1}{2}} P(x)\, dx = \sup\left\{ \int_{-1}^{+1} (1-x^2)^{-\frac{1}{2}} Q(x)\, dx : Q \in \mathbf{P}_{n-1}(f) \right\}$$

and so

$$\int_0^{\pi} p(\theta)\, d\theta = \sup\left\{ \int_0^{\pi} q(t)\, dt : q \in \mathbf{T}_{n-1}(h) \right\}$$

B. Next we consider the polynomial of best approximation to h from above. By Theorem 4, B, the polynomial P of best one sided approximation to f from above on $[-1, 1]$, of degree $\leqq n-1$, corresponding to the weight function $w(x) = (1-x^2)^{-\frac{1}{2}}$ is defined as follows:

If $n = 2l$, then

$$P(-1) = f(-1), \qquad P(1) = f(1), \qquad P(y_k) = f(y_k),$$
$$P'(y_k) = f'(y_k), \qquad k = 1, \ldots, l-1$$

where $y_1, \ldots, y_{l-1}$ are the zeros of the Tchebishev polynomial of the second kind U_{l-1}, i.e. $P_{l-1}^{(\frac{1}{2},\frac{1}{2})}$:

$$y_k = \cos\left(\frac{2k}{2l}\pi\right), \quad k = 1, \ldots, l-1 .$$

If $n = 2l+1$, then

$$P(1) = f(1), \qquad P(\eta_k) = f(\eta_k), \qquad P'(\eta_k) = f'(\eta_k), \qquad k = 1, \ldots, l$$

where $\eta_1, \ldots, \eta_l$ are the zeros of the Jacobi polynomial $P_l^{(\frac{1}{2},-\frac{1}{2})}$:

$$\eta_k = \cos\left(\frac{2k}{2l+1}\pi\right), \qquad k = 1, \ldots, l .$$

If we define $p(\theta) = P(\cos\theta)$, then p is the cosine polynomial of degree $\leqq n-1$ satisfying conditions (3.13) or (3.14) according to whether n is even or odd. Using the same argument as in part A, we find finally that p is the polynomial of best trigonometric approximation from above to h on $[0, \pi]$.

(iii) Our next theorem deals with polynomials of best one sided approximation to even functions on $[-1, 1]$. We shall consider the function f defined by

$$f(x) = h(x^2)$$

where h has derivatives of constant sign on $(0, 1)$. We shall assume here also that the weight function w is even. In that case, as it is well known, the polynomials (π_n), orthogonal on $[-1, 1]$ with respect to w, have symmetric zeros, i.e. if x_0 is a zero of π_n then $-x_0$ is also a zero of π_n, and all these zeros are in the interior of $[-1, 1]$.

THEOREM 6. *Suppose that h is continuous on* $[0, 1]$ *and that*

$$h^{([\frac{1}{2}n]+1)}(t) \geqq 0\,, \qquad t \in (0,1)\,. \tag{3.15}$$

Let f be defined on $[-1, 1]$ *by* $f(x) = h(x^2)$.

A. The polynomial P of best approximation to f from below on $[-1, 1]$, *of degree* $\leqq n$ *is defined by* $P(x) = Q(x^2)$, *where the polynomial Q is defined as follows:*

If $n = 4r-1$ *or* $n = 4r-2$, *then*

$$Q(t_\nu) = h(t_\nu)\,, \qquad Q'(t_\nu) = h'(t_\nu)\,, \qquad \nu = 1, \ldots, r \tag{3.16}$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r$ *are positive zeros of the polynomial* $\pi_{2r}^{(0,0)}$.
If $n = 4r$ *or* $n = 4r+1$, *then*

$$Q(0) = h(0)\,, \qquad Q(t_\nu) = h(t_\nu)\,, \qquad Q'(t_\nu) = h'(t_\nu)\,, \qquad \nu = 1, \ldots, r \tag{3.17}$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r$ *are positive zeros of the polynomial* $\pi_{2r+1}^{(0,0)}$.

B. The polynomial P of best approximation to f from above on $[-1, 1]$, *of degree* $\leqq n$, *is defined by* $P(x) = Q(x^2)$ *where the polynomial Q is defined as follows:*

If $n = 4r-1$ *or* $n = 4r-2$, *then*

$$(3.18) \qquad Q(0) = h(0), \quad Q(1) = h(1), \quad Q(t_\nu) = h(t_\nu),$$
$$Q'(t_\nu) = h'(t_\nu), \quad \nu = 1, \ldots, r-1$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r-1$ *are positive zeros of the polynomial* $\pi_{2r-1}^{(1,1)}$.
If $n = 4r$ *or* $n = 4r+1$, *then*

$$(3.19) \qquad Q(1) = h(1), \quad Q(t_\nu) = h(t_\nu), \quad Q'(t_\nu) = h'(t_\nu),$$
$$\nu = 1, \ldots, r$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r$ *are positive zeros of the polynomial* $\pi_{2r}^{(1,1)}$.

Remark. If

$$h^{([\frac{1}{2}n]+1)}(t) \leqq 0, \quad t \in (0,1)$$

then (3.16) and (3.17) and $P(x) = Q(x^2)$ define the polynomial P of best approximation *from above* to $f(x) = h(x^2)$, while (3.18), (3.19) and $P(x) = Q(x^2)$ define the polynomial P of best approximation *from below* to $f(x) = h(x^2)$.

Proof. A. Suppose first that $n = 4r-1$ or $n = 4r-2$. We have then $[\frac{1}{2}n] = 2r-1$. If Q is the polynomial defined by (3.16) we have $\deg Q \leqq 2r-1$ and by the remainder theorem and (3.15)

$$h(t) - Q(t) = \frac{h^{(2r)}(\tau)}{(2r)!}(t-t_1)^2 \ldots (t-t_r)^2 \geqq 0$$

i.e. $Q(t) \leqq h(t)$, $t \in [0, 1]$.
Let $P(x) = Q(x^2)$. Then $\deg P \leqq 2(2r-1) = 2[\frac{1}{2}n] \leqq n$ and

$$P(x) = Q(x^2) \leqq h(x^2) = f(x), \quad x \in [-1, 1].$$

Since the weight function w is even, the zeros of $\pi_{2r}^{(0,0)}$ are

$$x_1 = -\sqrt{t_r}, \ldots, x_r = -\sqrt{t_1}, x_{r+1} = \sqrt{t_1}, \ldots, x_{2r} = \sqrt{t_r}$$

and from the definition of Q follows that

$$P(x_\nu) = Q(x_\nu^2) = h(x_\nu^2) = f(x_\nu), \quad \nu = 1, \ldots, 2r.$$

Since $\deg P \leqq 4r-2 \leqq 4r-1$, from the quadrature formula (3.1) with $m = 2r$ follows that

$$\int_{-1}^{+1} w(x) P(x)\, dx = \sum_{\nu=1}^{2r} A_\nu^{2r} f(x_\nu).$$

On the other hand, since

$$n \leqq 2\left[\tfrac{1}{2}n\right] + 1 = 4r-1,$$

for any polynomial R of degree $\leqq n$ such that $R(x) \leqq f(x)$, $x \in [-1, 1]$, we have

$$\int_{-1}^{+1} w(x)\, R(x)\, dx \leqq \sum_{\nu=1}^{2r} A_\nu^{2r} f(x_\nu).$$

This proves the first part of the statement A.

If $n = 4r$ or $n = 4r+1$, then $[\frac{1}{2}n] = 2r$. If Q is the polynomial defined by (3.17), we have deg $Q \leqq 2r$ and by the remainder theorem and (3.15)

$$h(t) - Q(t) = \frac{h^{(2r+1)}(\tau)}{(2r+1)!}\, t\,(t-t_1)^2 \ldots (t-t_r)^2 \geqq 0$$

i.e. $Q(t) \leqq h(t)$, $t \in [0, 1]$.

Let $P(x) = Q(x^2)$. Then deg $P \leqq 4r = 2[\frac{1}{2}n] \leqq n$ and as before

$$P(x) \leqq f(x), \qquad x \in [-1, 1].$$

The zeros of $\pi_{2r+1}^{(0,0)}$ are

$$x_1 = -\sqrt{t_r}, \ldots, x_r = -\sqrt{t_1}, x_{r+1} = 0, x_{r+2} = \sqrt{t_1}, \ldots, x_{2r+1} = \sqrt{t_r}$$

and as in the preceding case it follows that $P(x_\nu) = f(x_\nu)$, $\nu = 1, \ldots, 2r+1$. Since deg $P \leqq 4r \leqq 2(2r+1) - 1$, using the quadrature formula (3.1) with $m = 2r+1$ we find that

$$\int_{-1}^{+1} w(t)\, P(t)\, dt = \sum_{\nu=1}^{2r+1} A_\nu^{2r+1} f(x_\nu).$$

On the other hand, for any polynomial R of degree $\leqq n$ such that $R(x) \leqq f(x)$, $x \in [-1, 1]$, we have

$$\int_{-1}^{+1} w(t)\, R(t)\, dt \leqq \sum_{\nu=1}^{2r+1} A_\nu^{2r+1} f(x_\nu).$$

This proves the second part of the statement A.

B. Consider first the case $n = 4r-1$ or $n = 4r-2$. We have then $[\frac{1}{2}n] = 2r-1$. If Q is the polynomial defined by (3.18) we have deg $Q \leqq 2r-1$ and by the remainder theorem and (3.15)

$$h(t) - Q(t) = \frac{h^{(2r)}(\tau)}{(2r)!} t(t-1)(t-t_1)^2 \dots (t-t_{r-1})^2 \leqq 0$$

i.e. $Q(t) \geqq h(t)$, $t \in [0, 1]$.

Let $P(x) = Q(x^2)$. Then $\deg P \leqq 2(2r-1) = 2[\frac{1}{2}n] \leqq n$ and $P(x) \geqq f(x)$, $x \in [-1, 1]$. Since the weight function is even, the zeros of $\pi_{2r-1}^{(1,1)}$ are

$$y_1 = -\sqrt{t_r}, \dots, y_{r-1} = -\sqrt{t_1}, \quad y_r = 0, \quad y_{r+1} = \sqrt{t_1}, \dots, y_{2r-1} = \sqrt{t_r}\,.$$

It follows that

$$P(-1) = f(-1), \quad P(1) = f(1), \quad P(y_\nu) = f(y_\nu), \quad \nu = 1, \dots, 2r-1\,.$$

Since $\deg P \leqq 4r-2 \leqq 4r-1$, using the quadrature formula (3.2) with $m = 2r$ we get

$$\int_{-1}^{+1} w(t) P(t)\, dt = B_0^{2r} f(-1) + B_{2r}^{2r} f(1) + \sum_{\nu=1}^{2r-1} B_\nu^{2r} f(y_\nu)\,.$$

On the other hand, since $n \leqq 2[\frac{1}{2}n]+1 = 4r-1$, for any polynomial R of degree $\leqq n$ such that $R(x) \geqq f(x)$, $x \in [-1, 1]$, we have

$$\int_{-1}^{+1} w(t) R(t)\, dt \geqq B_0^{2r} f(-1) + B_{2r}^{2r} f(1) + \sum_{\nu=1}^{2r-1} B_\nu^{2r} f(y_\nu)\,.$$

This proves the first part of the statement B.

Finally, if $n = 4r$ or $n = 4r+1$, then $[\frac{1}{2}n] = 2r$. If Q is the polynomial defined by (3.19), we have $\deg Q \leqq 2r$ and by the remainder theorem and (3.15)

$$h(t) - Q(t) = \frac{h^{(2r+1)}(\tau)}{(2r+1)!} (t-1)(t-t_1)^2 \dots (t-t_r)^2 \leqq 0$$

i.e. $Q(t) \geqq h(t)$, $t \in [0, 1]$.

Let $P(x) = Q(x^2)$. Then $\deg P \leqq 4r = 2[\frac{1}{2}n] \leqq n$ and $P(x) \geqq f(x)$, $x \in [-1, 1]$. If $y_1, \dots, y_{2r}$ are zeros of $\pi_{2r}^{(1,1)}$ we have

$$y_1 = -\sqrt{t_r}, \dots, y_r = -\sqrt{t_1}, y_{r+1} = \sqrt{t_1}, \dots, y_{2r} = \sqrt{t_r}$$

and so

$$P(-1) = f(-1), \quad P(1) = f(1), \quad P(y_\nu) = f(y_\nu),$$
$$\nu = 1, \ldots, 2r.$$

Since $\deg P \leqq 4r \leqq 4r+1 = 2(2r+1) - 1$, using quadrature formula (3.2) with $m = 2r+1$ we get

$$\int_{-1}^{+1} w(t) P(t)\, dt = B_{2r+1}^{2r+1} f(-1) + B_{2r+1}^{2r+1} f(1) + \sum_{\nu=1}^{2r} B_\nu^{2r+1} f(y_\nu).$$

On the other hand, since $n \leqq 2[\frac{1}{2}n]+1 = 4r+1 = 2(2r+1) - 1$, for any polynomial R of degree $\leqq n$ such that $R(x) \geqq f(x)$, $x \in [-1, 1]$, we have

$$\int_{-1}^{+1} w(t) R(t)\, dt \geqq B_0^{2r+1} f(-1) + B_{2r+1}^{2r+1} f(1) + \sum_{\nu=1}^{2r} B_\nu^{2r+1} f(y_\nu).$$

This proves the second part of the statement B.

(iv) Finally we shall mention explicitly a special case of the preceding theorem, corresponding to the function $h(t) = \sqrt{t}$. We shall assume again that the weight function w is even.

Observing that $h^{(2m)}(t) < 0$ and $h^{(2m-1)}(t) > 0$, $t \in (0,1)$, and taking into account the remark following Theorem 6, we obtain immediately the following results:

A. The polynomial P of best approximation to $f(x) = |x|$ *from below* on $[-1, 1]$ is defined by $P(x) = Q(x^2)$, where the polynomial Q is defined as follows:

If $n = 4r-1$ or $n = 4r-2$, then

$$Q(0) = 0, \quad Q(1) = 1, \quad Q(t_\nu) = \sqrt{t_\nu}, \quad Q'(t_\nu) = \frac{1}{2\sqrt{t_\nu}},$$
$$\nu = 1, \ldots, r-1$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r-1$ are positive zeros of $\pi_{2r-1}^{(1,1)}$.

If $n = 4r$ or $n = 4r+1$, then

$$Q(0) = 0, \quad Q(t_\nu) = \sqrt{t_\nu}, \quad Q'(t_\nu) = \frac{1}{2\sqrt{t_\nu}}, \quad \nu = 1, \ldots, r$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r$ are positive zeros of $\pi_{2r+1}^{(0,0)}$.

B. The polynomial P of best approximation to $f(x) = |x|$ *from above* on $[-1, 1]$ of degree $\leq n$ is defined by $P(x) = Q(x^2)$, where the polynomial Q is defined as follows:

If $n = 4r-1$ or $n = 4r-2$, then

$$Q(t_\nu) = \sqrt{t_\nu}\,, \qquad Q'(t_\nu) = \frac{1}{2\sqrt{t_\nu}}\,, \qquad \nu = 1, \ldots, r$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r$ are positive zeros of $\pi_{2r}^{(0,0)}$.

If $n = 4r$ or $n = 4r+1$, then

$$Q(1) = 1\,, \qquad Q(t_\nu) = \sqrt{t_\nu}\,, \qquad Q'(t_\nu) = \frac{1}{2\sqrt{t_\nu}}\,, \qquad \nu = 1, \ldots, r$$

where $\sqrt{t_\nu}$, $\nu = 1, \ldots, r$ are positive zeros of $\pi_{2r}^{(1,1)}$.

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