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ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT

by Rudolph E. LANGER

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1. INTRODUCTION.

A common type of linear ordinary differential equation—indeed the one which includes the great majority of such equations that figure prominently in applications, and for which literatures therefore exist—is

$$\frac{d^n u}{dz^n} + \lambda p_1(z, \lambda) \frac{d^{n-1} u}{dz^{n-1}} + \ldots + \lambda^n p_n(z, \lambda) u = 0 , \qquad (1.1)$$

in which λ is a parameter, and the coefficients $p_j(z, \lambda)$ are either:

(a) free from λ;
(b) polynomials in $1/\lambda$;
(c) analytic functions of $1/\lambda$ in a domain $|\lambda| \geqq M > 0$;
(d) functions that can be asymptotically represented by power series in $1/\lambda$ in a sector of the complex λ plane.

This paper is concerned with equations of this type for large values of $|\lambda|$.

The consideration of any differential equation— unless its coefficients be constants—must be relevant to a domain of the variable. The objective of this paper can be made clear only after some description of this domain has been given.

An equation (1.1) has associated with it the so-called *auxiliary equation*

$$\chi^n + p_1(z, \infty) \chi^{n-1} + \ldots + p_n(z, \infty) = 0 . \qquad (1.2)$$

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As a polynomial equation in χ, this has roots which are functions of z. In a given z-region, if z is complex, or on a given interval, if z is real, these *auxiliary roots* may all be simple or various multiplicities may occur among them. A multiplicity may be permanent in z over the region, or over a sub-region, or, on the other hand, it may be isolated, in which case it maintains at a point without doing so at other points of the neighborhood. An isolated point of multiplicity is called a *turning-point.*

Fully developed theory is extant, and can justifiably be referred to as classical, for the determination of the asymptotic forms of the solutions of a differential equation (1. 1) over any closed z-region which completely excludes turning-points. This theory applies, of course, irrespective of the region, to all equations (1. 1) with constant coefficients. The state of the theory is very different, namely quite fragmentary, when a turning-point is lodged within the region. For this reason, and also because modern physical theories require it, the study of the solution forms of an equation (1. 1) in a region about a turning-point is of emminent contemporary interest. The classical algorithms fail irretrievably in such a region, a fact which has been shown to be inevitable by results otherwise obtained, because the forms yielded by those algorithims lack adequacy to reflect the intricate functional metamorphoses which characterize the solutions of the differential equation in a turning-point neighborhood. The profundity of these changes is suggested by even so simple an example as the differential equation

$$\frac{d^2 u}{dx^2} + \lambda^2 xu = 0,$$

with x and λ^2 real. The origin is in this case a turning point, and about this point the solutions undergo transitions between oscillatory and exponential function types.

A turning-point is to be characterized in the first instance by the configuration of the auxiliary roots in its neighborhood. In the case of a differential equation of the second order, for which there are just two auxiliary roots, $\chi_1(z)$, $\chi_2(z)$, the grounds for distinction are limited, being in fact restricted to the degree to which the root difference $(\chi_1 - \chi_2)$ vanishes. When equations of higher and higher orders are taken into consideration, the grounds for distinction rapidly multiply. Between the possible extreme configurations in which the roots all coincide with each other, and in which only two roots come into a coincidence, there are all the intermediate ones in which certain roots may come into one coincidence, while other roots come into other coincidences quite apart. A simple example of this kind is afforded by the differential equation

$$\frac{d^4 u}{dz^4} - 2\lambda \frac{d^3 u}{dz^3} + \lambda^2 \left(1 - \frac{1}{\lambda}\right) \frac{d^2 u}{dz^2} - 2z\lambda^3 \frac{du}{dz}$$

$$+ \lambda^4 \left[z - z^2 + \frac{z-1}{\lambda} \right] u = 0 . \tag{1.3}$$

The auxiliary roots of this are $\chi_1 = iz^{\frac{1}{2}}$, $\chi_2 = -iz^{\frac{1}{2}}$, $\chi_3 = 1 + z^{\frac{1}{2}}$, $\chi_4 = 1 - z^{\frac{1}{2}}$.
At the turning-point $z = 0$, $\chi_1 = \chi_2$ and $\chi_3 = \chi_4$, but $\chi_1 \neq \chi_3$.

Algebraically the characteristics of an auxiliary root configuration are, of course, expressible in terms of the reducibility of the auxiliary equation (1. 2). In these terms the objective of the present paper can now be explained. A methodical key to the deduction of the asymptotic solution forms of a given differential equation has been brought to hand when a so-called *related equation* has been found. The term *related equation* in this context signifies another differential equation which

(a) has coefficients that are the same as those of the given equation out to terms of an arbitrarily pre-specified degree in $1/\lambda$, and

(b) whose solution forms are known.

The analysis through which such an equation may be applied is not included in this paper, mainly because it is essentially systematic and has been set forth in a number of instances in the literature [1], [2], [3], [4], [5], [6]. The finding, or the construction, of a related equation is not, and could not be expected to be, systematic. It is in this that ingenuity is an essential requisite. As might well be expected, its difficulty mounts rapidly with the order of the differential equation, and therefore any means for referring the problem from a given equation to ones of lower order are treasurable. This brings us to the point. In an earlier proper [5] I have shown that if, in a given region, p of the auxiliary roots of a differential equation (1. 1) are simple, the construction of a related equation is referrable to such a construction for an equation of the lower order $(n-p)$. The present paper goes materially further. It shows that if the auxiliary polynomial in χ factors into relatively prime factors of the degrees p and q, the coefficients of which are analytic, the construction of a related equation is referrable to such constructions for differential equations of the lower orders p and q. It will be seen at once, by iteration, that if the auxiliary polynomial has analytic factors of the degrees $p_1, p_2, \dots p_k$, the construction of a related equation is referrable to such constructions for differential equations of the orders $p_1, p_2, \dots p_k$. The equation (1. 3) serves again as an example. Its auxiliary equation may be written

$$(\chi^2 + z)(\chi^2 - 2\chi + 1 - z) = 0 .$$

A related equation for it can be constructed, because that can be done for the two equations of lower order

$$\frac{d^2 v}{dz^2} + \lambda^2 \left(z - \frac{1}{\lambda}\right) v = 0 ,$$

$$\frac{d^2 y}{dz^2} - 2\lambda \frac{dy}{dz} + \lambda^2 (1 - z) y = 0 .$$

2. The hypotheses.

The given differential equation (1. 1) may be conveniently denoted by

$$L(u) = 0 , \tag{2. 1}$$

with

$$L(u) \equiv \sum_{j=0}^{n} \lambda^j p_j(z, \lambda) D^{n-j} u , \quad p_0 \equiv 1 , \tag{2. 2}$$

the symbol $D^k f$ signifying $d^k f/dz^k$. It is an overall objective of this discussion to describe the construction of another differential equation subject to the specification, among others, that its coefficients be the same as those of (2. 2) to the extent of all terms which, in terms of $1/\lambda$, are of a degree less than a certain arbitrarily prescribed one. We shall designate this prescribed degree as the r^{th}. The domains of the variable z and the parameter λ shall be respectively a closed region of the complex z-plane and any portion of the complex λ-plane in which $|\lambda|$ is bounded below by a positive bound but is unbounded above, and in which the coefficients of (2. 2) have the forms

$$p_j(z, \lambda) = \sum_{\mu=0}^{r-1} \frac{p_{j,\mu}(z)}{\lambda^\mu} + \frac{p_{j,r}(z, \lambda)}{\lambda^r}, \quad j = 1, 2, \ldots, n, \qquad (2.3)$$

with each $p_{j,\mu}(z)$, $\mu < r$ analytic, and $p_{j,r}(z, \lambda)$ bounded.

The auxiliary equation (1. 2) is accordingly

$$P(\chi) = 0, \qquad (2.4)$$

with

$$P(\chi) = \sum_{j=0}^{n} p_{j,0}(z)\, \chi^{n-j}. \qquad (2.5)$$

It shall be a hypothesis that in the z-region under consideration the polynomial $P(\chi)$ factors, thus

$$P(\chi) = B(\chi)\, \Gamma(\chi), \qquad (2.6)$$

the factors

$$\begin{aligned} B(\chi) &= \sum_{j=0}^{p} b_j(z)\, \chi^{p-j}, \quad b_0 \equiv 1, \\ \Gamma(\chi) &= \sum_{j=0}^{q} c_j(z)\, \chi_q^{q-j}, \quad c_0 \equiv 1, \end{aligned} \qquad (2.7)$$

being relatively prime, and having analytic coefficients. The z-region shall contain no more than one turning-point, and if there is such a point it shall be taken to be the origin.

The relations (2. 5), (2. 6) and (2. 7) imply that $p+q=n$. We shall suppose the notation to be so assigned as to yield $p \leqq q$. It follows then also that

$$\sum_{i=0}^{k} b_i(z)\, c_{k-i}(z) = p_{k,\,0}(z)\,, \quad k = 0,\ 1,\ 2,\ \dots,\ n\,, \qquad (2.\ 8)$$

it being understood that the value 0 is to be assigned to any symbol $b_i(z)$ or $c_{k-i}(z)$ which, by virtue of its subscript, is not present in the formulas (2. 7).

A change of dependent variable can be applied to the equation (1. 1) to give it a form for which the coefficient $b_1(z)$ is identically zero. Considerable simplifications of the formulas result therefrom. We shall not resort to that normalization, however, refraining from it in order to keep the roles of the factors $B(\chi)$ and $\Gamma(\chi)$ interchangeable.

3. The resultant.

It is a hypothesis that the two polynomials (2. 7) are relatively prime over the z-region. If we denote their resultant by $\Delta(z)$, we have accordingly

$$\Delta(z) \neq 0\,, \qquad (3.\ 1)$$

with

$$\Delta(z) \equiv \begin{vmatrix} 1 & b_1 & b_2 & - & - & - & - & b_p & 0 & - & - & - & - & - & - & 0 \\ 0 & 1 & b_1 & b_2 & - & - & - & - & b_p & 0 & - & - & - & - & - & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & - & - & - & - & - & - & - & 0 & 1 & b_1 & - & - & - & - & b_p \\ 1 & c_1 & c_2 & - & - & - & - & - & - & - & - & c_q & 0 & - & - & 0 \\ 0 & 1 & c_1 & - & - & - & - & - & - & - & - & - & c_q & 0 & - & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & - & - & - & - & 0 & 1 & c_1 & c_2 & - & - & - & - & - & - & c_q \end{vmatrix} \qquad (3.\ 2)$$

We shall find use for the relation (3. 2). However, for future use we find it convenient to formulate the relative primary of the factors $B(\chi)$ and $\Gamma(\chi)$ also in an alternative form. This is done as follows:

Let K designate the familiar matrix

$$K(z) = \begin{bmatrix} 0 & 0 & - & - & - & - & -b_p \\ 1 & 0 & - & - & - & - & -b_{p-1} \\ 0 & 1 & 0 & - & - & - & -b_{p-2} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & - & - & - & - & 1 & -b_1 \end{bmatrix} \tag{3. 3}$$

The eigen-values of this are the roots x_i, $i = 1, 2, \ldots, p$ of the equation $B(x) = 0$. If we designate by ξ_i an eigen-vector corresponding to x_i we have

$$K^h \xi_i = x_i^h \xi_i, \quad h = 1, 2, \ldots, q,$$

and accordingly

$$\Gamma(K)\xi_i = \Gamma(x_i)\xi_i .$$

Thus $\Gamma(x_i)$ is an eigenvalue of the matrix $\Gamma(K)$, and, since the product of the eigenvalues is the determinant of the matrix, we have

$$|\Gamma(K)| = \prod_{i=1}^{p} \Gamma(x_i) .$$

Observing that no factor on the right is zero, and giving to $\Gamma(K)$ its explicit form, we conclude with the result

$$\left|\sum_{j=0}^{q} c_j(z) K^{q-j}(z)\right| \neq 0 \tag{3. 4}$$

4. Two differencial operators of the orders p and q.

Let the functions $\beta_j(z, \lambda)$ and $\gamma_i(z, \lambda)$ be taken to be polynomials of the degree $(r - 1)$ in $1/\lambda$, thus

$$\begin{aligned} \beta_j(z, \lambda) &= \sum_{\nu=0}^{r-1} \frac{\beta_{j,\nu}(z)}{\lambda^\nu}, \quad \beta_{j,0}(z) \equiv b_j(z); \; j = 1, 2, \ldots, p, \\ \gamma_i(z, \lambda) &= \sum_{\nu=0}^{r-1} \frac{\gamma_{i,\nu}(z)}{\lambda^\nu}, \quad \gamma_{i,0}(z) \equiv c_i(z); \; i = 1, 2, \ldots, q . \end{aligned} \tag{4. 1}$$

As has been indicated, the terms of the zeroth degree are to be the coefficients which appear in the formulas (2. 7). The remaining terms, $\beta_{j,\nu}(z)$ and $\gamma_{i,\nu}(z)$, with $\nu \geqq 1$, shall be analytic over the z-region, but beyond that shall be left, for the moment, unspecified. By l and m we shall designate the differential operators

$$
\begin{aligned}
l &= \sum_{j=0}^{p} \lambda^j \beta_j(z,\lambda) D^{p-j}, \quad \beta_0 \equiv 1, \\
m &= \sum_{i=0}^{q} \lambda\, \gamma_i(z,\lambda) D^{q-i}, \quad \gamma_0 \equiv 1.
\end{aligned}
\tag{4. 2}
$$

The immediate objective will be to show that the unspecified terms in the formulas (4. 1) can be so chosen as to give the differential form $l(m(u))$ coefficients which differ from those of the form (2. 2) only by terms that are of at least the r^{th} degree in $1/\lambda$.

The k-fold differentiation of $m(y)$ yields the formula

$$
D^k m(y) = \sum_{i=0}^{q} \sum_{s=0}^{k} \lambda^i \binom{k}{s} D^s \gamma_i D^{q-i+k-s} y,
$$

in which the symbol $\binom{k}{s}$ denotes, as customarily, the coefficient of x^s in the binomial expansion of $(1+x)^k$. On using $i+s$ in place of i as the variable of summation, and observing that the terms which appear to have been gratuitously included are ones to which the value zero are to be assigned, we find that the formula may be written

$$
D^k m(y) = \sum_{i=0}^{q+k} \sum_{s=0}^{p} \lambda^{i-s} \binom{k}{s} D^s \gamma_{i-s} D^{q+k-i} y, \qquad k = 0, 1, 2, \ldots, p. \tag{4. 3}
$$

From this it follows that

$$
l(m(y)) = \sum_{j=0}^{p} \sum_{i=0}^{q} \sum_{s=0}^{p-j} \lambda^{j+i-s} \binom{p-j}{s} \beta_j D^s \gamma_{i-s} D^{n-j-i} y.
$$

This formula is again improved by using $i+j$ in place of i as the variable of summation. It becomes, then

$$l(m(y)) = \sum_{i=0}^{n} \lambda^i \Psi_i(z, \lambda) D^{n-i} y, \tag{4.4}$$

with

$$\Psi_i(z, \lambda) = \sum_{j=0}^{p} \sum_{s=0}^{p-j} \lambda^{-s} \binom{p-j}{s} \beta_j D^s \gamma_{i-j-s} \,. \tag{4.5}$$

The functions $\Psi_i(z, \lambda)$, inasmuch as they are combinations of those given in (4. 1), are polynomials in $1/\lambda$. We may therefore write them in the form

$$\Psi_i(z, \lambda) = \sum_{\mu=0}^{r-1} \frac{\psi_{i,\mu}(z)}{\lambda^\mu} + \frac{\psi_{i,r}(z, \lambda)}{\lambda^r} \,. \tag{4.6}$$

A comparison of the terms in like powers of $1/\lambda$ in the relations (4. 5) and (4. 6) yields formulas for the functions $\Psi_{i,\mu}(z)$. Those for which $\mu = 0$ are particularly easy to obtain. On setting $s = 0$ in (4. 5), and replacing β_j and γ_{i-j} by their leading terms b_j and c_{i-j}, we find that

$$\psi_{i,0}(z) = \sum_{j=0}^{p} b_j(z) c_{i-j}(z) \,.$$

Recourse to the relation (2. 8) thus shows that

$$\psi_{i,0}(z) = p_{i,0}(z) \,, \quad i = 1, 2, \ldots, n \,. \tag{4.7}$$

At least to the extent of the leading terms of their coefficients, the forms (2. 2) and (4. 4) are, therefore, the same.

5. A determination of unspecified coefficients.

We propose now to deduce a formula for the general coefficient $\psi_{i,\mu}(z)$ in (4. 6) by selecting the multiplier of the appropriate power of $1/\lambda$ from the formula (4. 5). To begin with, it follows from the relations (4. 1) that

$$\beta_j D^s \gamma_{i-j-s} = \sum_{\mu=0}^{2r-2} \sum_{k=0}^{\mu} \lambda^{-\mu} \beta_{j,k} D^s \gamma_{i-j-s,\mu-k}$$

By virtue of this, the relation (4. 5) may be more precisely written as

$$\psi_i(z, \lambda) = \sum_{j=0}^{p} \sum_{s=0}^{p-j} \sum_{\mu=0}^{2r-2} \sum_{k=0}^{\mu} \lambda^{-s-\mu} \binom{p-j}{s} \beta_{j,k} D^s \gamma_{i-j-s,\,\mu-k} \,.$$

By the use of $\mu+s$ as a variable of summation in place of μ, this is, however, seen to take the form (4. 6) with

$$\psi_{i,\mu}(z) = \sum_{j=0}^{p} \sum_{s=0}^{p-j} \sum_{k=0}^{\mu=s} \binom{p-j}{s} \beta_{j,k} D^s \gamma_{i-j-s,\,\mu-s-k} \,. \tag{5. 1}$$

An inspection of this result reveals an important fact, namely, that the functions $\psi_{i,\mu}(z)$, with any specific μ, do not depend at all upon any of the elements $\beta_{j,i}(z)$, $\gamma_{j,i}(z)$ for which $i > \mu$. Moreover these elements with $i = \mu$ are involved in them precisely to the respective extent

$$\sum_{j=0}^{p} \{ \beta_{j,\mu} \gamma_{i-j,0} + \beta_{j,0} \gamma_{i-j,\mu} \} ,$$

namely, on dropping the terms to which the value zero must be assigned, reversing one of the summations, and recalling that $\beta_{j,0} = b_j$ and $\gamma_{j,0} = c_j$, to the extent

$$\sum_{j=0}^{i} \{ b_{i-j} \gamma_{j,\mu} + c_{i-j} \beta_{j,\mu} \} \,.$$

The formulas (5. 1) therefore have the form

$$\psi_{i,\mu}(z) = \sum_{j=l}^{i} \{ b_{i-j} \gamma_{j,\mu} + c_{i-j} \beta_{j,\mu} \} + \varphi_{i,\mu}(z) \,, \tag{5. 2}$$

with $\varphi_{i,\mu}(z)$ denoting a function which is contructed of the elements $\beta_{j,i}$ and $\gamma_{j,i}$ in which $i < \mu$.

We recall now that the elements $\beta_{j,i}(z)$, $\gamma_{j,i}(z)$ with $i \geqq 1$ were left unspecified, except that they be analytic, and inquire whether they may be so specified as to make the formulas (5. 2) yield assigned functions. The particular assignment envisaged is

$$\psi_{i,\mu}(z) = p_{i,\mu}(z) \,, \; i = 1, 2, \dots, n \,; \; \mu = 1, 2, \dots, (r-1) \,. \tag{5. 3}$$

This question is, in other terms, whether the equations

$$\sum_{j=1}^{i}\{b_{i,j}\gamma_{j,\mu}+c_{i-j}\beta_{j,\mu}\} = p_{i,\mu}(z)-\varphi_{i,\mu}(z), \quad \begin{array}{l} i = 1, 2, \ldots, n; \\ \mu = 1, 2, \ldots, (r-1), \end{array} \tag{5.4}$$

can be fulfilled by choice of the functions $\beta_j(z, \lambda)$, $\gamma_j(z, \lambda)$ of (4. 1).

Consider first the case in which $\mu = 1$. In this case the right-hand members of the equations are known, since the functions $\varphi_{i,1}(z)$ are made up of the known elements $b_j(z)$, $c_j(z)$. The equations therefore comprise a linear non-homogeneous systems in the " unknowns " $\gamma_{1,1} \cdots \gamma_{q,1}$, $\beta_{1,1}, \cdots \beta_{p,1}$, and the determinant of this system is seen to be $\Delta(z)$, the determinant (3. 2), written with rows and columns interchanged. Since this is nowhere zero in the z-region, by (3. 1), the system is analytically solvable, and by the solution the equations (5. 3) for $\mu = 1$ are assured.

We proceed now by induction. Assuming that the elements $\beta_{j,i}$, $\gamma_{j,i}$ have been determined for $i = 1, 2, \ldots, (\mu - 1)$, we consider the system (5. 4). The right-hand members of the equations are known, and the determinant of the system is $\Delta(z)$. The system is, therefore, analytically solvable for $\gamma_{1,\mu}, \cdots \gamma_{q,\mu}$, $\beta_{1,\mu}, \cdots \beta_{p,\mu}$, for successive values of μ. By these solutions the equations (5. 3) are fulfilled, and now, from a comparison of the formula (2. 2) with (4. 4), and of (2. 3) with (4. 6) and (5. 3), we see that

$$L(u) = l(m(u))+\frac{1}{\lambda^r}\sum_{j=1}^{n}\lambda^j\{p_{j,r}(z, \lambda)-\psi_{j,r}(z, \lambda)\}D^{n-j}u. \tag{5.5}$$

The differential operators (4. 2), and therewith the differential equations

$$\begin{aligned} l\,(v) &= 0, \\ m(y) &= 0, \end{aligned} \tag{5.6}$$

are now completely specific.

6. A set of functions $\eta_i(z, \lambda)$.

It is the immediate purpose of this paper to show that under the hypotheses that have been made the construction of a related equation for the differential equation (2. 1) is referable to such constructions for equations of lower order. These latter will be precisely the differential equations (5. 6). In proceeding, we shall therefore suppose that the related equations

$$l^*(v) = 0, \tag{6. 1}$$

with

$$l^*(v) \equiv \sum_{j=0}^{p} \lambda^j \bar{\beta}_j(z, \lambda) D^{p-j} v, \tag{6. 2}$$

and

$$m^*(y) = 0, \tag{6. 3}$$

with

$$m^*(y) \equiv \sum_{i=0}^{q} \lambda^i \bar{\gamma}_i(z, \lambda) D^{q-i} y \tag{6. 4}$$

have been brought to hand. Their coefficients are of the forms

$$\bar{\beta}_j(z, \lambda) = \beta_j(z, \lambda) + \frac{1}{\lambda^r} \bar{\beta}_{j,r}(z, \lambda),$$

$$\bar{\gamma}_j(z, \lambda) = \gamma_j(z, \lambda) + \frac{1}{\lambda^r} \bar{\gamma}_{j,r}(z, \lambda), \tag{6. 5}$$

the functions $\bar{\beta}_{i,r}$ and $\bar{\gamma}_{j,r}$ being bounded over the domains of z and λ. Since terms of a degree greater than the $(r-1)^{th}$ in $1/\lambda$ were never involved in the derivation of the relation (5. 5), it is clear that also

$$L(u) = l^*(m^*(u)) + \frac{1}{\lambda^r} \sum_{j=1}^{n} \lambda^j \epsilon_j(z, \lambda) D^{n-j} u, \tag{6. 6}$$

with certain bounded coefficients ε_j.

As for all related equations, by definition, the solutions of the equations (6. 1) and (6. 3) are known. We shall denote complete sets for these solutions respectively by $v_i(z, \lambda)$, $i = 1, 2, \ldots\ p$; and $y_j(z, \lambda)$, $j = 1, 2, \ldots, q$. Our method, as will be seen, requires a modification of one of these solution sets. That is the purpose of the replacement of the functions $v_i(z, \lambda)$ by corresponding ones $\eta_i(z, \lambda)$ to which we proceed.

Let the functions $a_j^{(o)}(z, \lambda)$ be polynomials of the degree $(r-1)$ in $1/\lambda$, namely

$$\alpha_j^{(0)}(z, \lambda) = \sum_{\nu=0}^{r-1} \frac{\alpha_{j,\nu}^{(0)}(z)}{\lambda^\nu}, \tag{6.7}$$

with coefficients $a_{j,\nu}^{(0)}(z)$ that are analytic but, for the time being, unspecified. Then consider the formulas

$$\eta_i(z, \lambda) = \sum_{j=1}^{p} \lambda^{1-j} \alpha_j^{(0)}(z, \lambda) D^{j-1} v_i(z, \lambda), \quad i = 1, 2, \ldots, p. \tag{6.8}$$

The differentiation of this, and the subsequent elimination of $D^p v_i$ by use of the equation (6. 1), yields a corresponding formula for $D\eta_i$, and repetitions of the procedure yield more generally that

$$D^k \eta_i = \lambda^k \sum_{j=1}^{p} \lambda^{i-j} \alpha_j^{(k)}(z, \lambda) D^{j-1} v_i(z, \lambda), \quad k = 0, 1, 2, \ldots, q. \tag{6.9}$$

In this the coefficients are recursively given by the relation

$$\alpha_j^{(k)} = \alpha_{j-1}^{(k-1)} - \bar{\beta}_{p+1-j}\, \alpha_p^{(k-1)} + \frac{1}{\lambda} D\alpha_j^{(k-1)} \tag{6.10}$$

The relations (6. 9) yield the formula

$$m^*(\eta_i) = \lambda^q \sum_{j=1}^{p} \lambda^{1-j} \sigma_j^{(0)}(z, \lambda) D^{j-1} v_i. \tag{6.11}$$

with

$$\sigma_j^{(0)}(z, \lambda) = \sum_{k=0}^{q} \gamma_k \alpha_j^{(q-k)}. \tag{6.12}$$

These latter coefficients $\sigma_j^{(0)}(z, \lambda)$ are evidently expressible in powers of $1/\lambda$, and are therefore of the form

$$\sigma_j^{(0)}(z, \lambda) \sum_{\nu=0}^{r-1} \frac{\sigma_{j,\nu}^{(0)}(z)}{\lambda^\nu} + \frac{\sigma_{j,r}^{(0)}(z, \lambda)}{\lambda^r} \tag{6.13}$$

with each $\sigma_{j,\nu}^{(0)}(z)$ analytic, and $\sigma_{j,r}^{(0)}(z,\lambda)$ bounded. We shall show that the elements $\alpha_{j,\nu}^{(0)}(z)$ in (6. 7) may be so specified as to yield

$$\sigma_{j,\nu}^{(0)}(z) \equiv \begin{cases} 1 \text{ when } (j,\nu) = (1,0), \quad \nu = 0, 1, 2, \ldots, (r-1) \\ 0 \text{ when } (j,\nu) \neq (1,0). \end{cases} \tag{6, 14}$$

The effect of this will be to give the formula (6. 11) the form

$$m^*(\eta_i) = \lambda^q \left\{ v_i(z,\lambda) + \frac{1}{\lambda^r} \sum_{j=1}^{p} \lambda^{1-j} \sigma_{j,r}^{(0)}(z,\lambda) D^{j-1} v_i \right\}. \tag{6. 15}$$

7. Another determination of coefficients.

The dependence of the functions (6. 12) upon the unspecified ones $\alpha_{j,\nu}^{(0)}(z)$ of (6. 7) is advantageously set forth in terms of vector-matrix notation. To this end, let a column vector with the components φ_i, $i = 1, 2, \ldots, p$, be denoted by (φ) and let the vector whose components are the terms in $1/\lambda^\nu$ of (φ), namely with the components $\varphi_{i,\nu}$ $i = 1, 2, \ldots, p$, be denoted by $(\varphi)_\nu$. Also let H designate the square matrix

$$H = \begin{bmatrix} \lambda^{-1}D & 0 & 0 & - & - & -\bar{\beta}_p \\ 1 & \lambda^{-1}D & 0 & - & - & -\bar{\beta}_{p-1} \\ 0 & 1 & \lambda^{-1}D & - & - & -\bar{\beta}_{p-2} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & - & - & - & -\bar{\beta}_1 + \lambda^{-1}D \end{bmatrix} \tag{7. 1}$$

the elements of which are in part functions of z and λ, and in part the indicated differential operator. Again let H_ν designate the matrix that is obtainable from (7. 1) by replacing its elements by their terms in $1/\lambda^\nu$. The relations (6. 10) are then seen at once to take the form

$$(\alpha^{(k)}) = H(\alpha^{(k-1)}).$$

With iteration defined in the manner

$$H^{[k]}(\varphi) = H(H^{[k-1]}(\varphi)), \; H^{[1]} = H, \tag{7. 2}$$

and with $H^{[0]}$ signifying the unit matrix, it is then easily seen that

$$(\alpha^{(k)}) = H^{[k]}(\alpha^{(0)}). \tag{7.3}$$

The relation (6. 12) may thus be written in the form

$$(\sigma^{(0)}) = J(\alpha^{(0)}), \tag{7.4}$$

with J standing for the matrix

$$J = \sum_{k=0}^{q} \bar{\gamma}_k H^{[q-k]}. \tag{7.5}$$

The evaluations

$$(\sigma^{(0)})_\nu = \sum_{j=0}^{\nu} J_j (\alpha^{(0)})_{\nu-j},$$

$$J_j = \sum_{k=0}^{q} \sum_{i=0}^{j} \bar{\gamma}_{k,i} H_{j-i}^{[q-k]},$$

evidently combine to yield the formula

$$(\sigma^{(0)})_\nu = \sum_{k=0}^{q} \sum_{j=0}^{\nu} \sum_{i=0}^{j} \bar{\gamma}_{k,i} H_{j-i}^{[q-k]} (\alpha^{(0)})_{\nu-j}. \tag{7.6}$$

In connection with this, certain observations are apropos. To begin with, the index value $j = 0$ implies $i = 0$, whereas by (6. 5) and (4. 1), $\bar{\gamma}_{k,0} = c_k(z)$. Further when $i = j$ the matrix $H_{j-i}^{[q-k]}$ reduces to precisely $K^{q-k}(z)$, with $K(z)$ as given in (3. 3). On the basis of these facts the equation (7. 6) may be arranged into the form

$$\sum_{k-0}^{q} c_k(z) K^{q-k}(z)(\alpha^{(0)})_\nu = (\sigma^{(0)})_\nu - \sum_{k=0}^{q} \sum_{j=1}^{\nu} \sum_{i=0}^{j} \bar{\gamma}_{k,i} H_{j-i}^{[q-k]} (\alpha^{(0)})_{\nu-j} \tag{7.7}$$

This is a vector equation for $(a^{(0)})_\nu$, which we shall consider for successive values of ν, assuming that the values (6. 14) have been assigned.

When $\nu = 0$, the triple sum on the right of the equality in (7. 7) vanishes, and the right-hand member is, therefore, the vector $(\sigma^{(0)})_0$ whose first component is 1 and whose other components are 0. The equation is therefore a non-homogeneous one, and accordingly admits of an analytic solution for $(a^{(0)})_0$ provided the matrix multiplier of this vector on the left is non-singular. This condition is assured by the relation (3. 4).

Now we may proceed by induction. Assuming that the vectors $(a^{(0)})_j$ for $j = 1, 2, \ldots, (\nu-1)$, have been determined and are analytic, the right-hand member of the equation (7. 7) is known. As in the case $\nu = 0$, so now, the equation is analytically solvable. The solutions for the successive values $\nu = 0, 1, 2, \ldots, (r-1)$, yield the coefficients (6. 7) for which the functions $\eta_i(z, \lambda)$, as given by the formulas (6. 8), fulfill the relations (6. 5).

8. On linear independence.

With the functions $a_j^{(0)}(z, \lambda)$ now at hand, we have at our disposal the n known functions $y_j(z, \lambda)$, $j = 1, 2, \ldots, q$, which are the solutions of the differential equation (6. 3), and $\eta_i(z, \lambda)$, $i = 1, 2, \ldots, p$, which are given by the formulas (6. 8). We shall show that these functions are linearly independent.

Let the Wronskians of the entire set and of the respective sub-sets be denoted respectively by W_n, $W_q(y)$ and $W_p(\eta)$. If the usual form

$$W_n = \begin{bmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_q \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{n-1}y_1 & - & - & - & D^{n-1}y_q & D^{n-1}\eta_1 & - & - & - & D^{n-1}\eta_p \end{bmatrix} \tag{8. 1}$$

is modified by adding to each of the last p rows suitable multiples of the preceding ones, the formula can be made to appear thus

$$= \begin{bmatrix} y_1 & - & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - & - \\ D^{q-1}y_1 & - & - & - & - & D^{q-1}y_q & D^{q-1}\eta_1 & - & - & - & D^{q-1}\eta_p \\ m^*(y_1) & - & - & - & - & m^*(y_q) & m^*(\eta_1) & - & - & - & m^*(\eta_p) \\ Dm^*(y_1) & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - \\ D^{p-1}m^*(y_1) & - & - & - & - & D^{p-1}m^*(y_q) & D^{p-1}m^*(\eta_1) & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix} \tag{8. 2}$$

In this, however, each of the elements occupying a position in one of the first q columns and in one of the last p rows is zero. The formula therefore reduces at once to

$$W_n = W_q(y)\,T, \tag{8.3}$$

with

$$T = \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix}. \tag{8.4}$$

Now $m^*(\eta_j)$ is given by the formula (6. 15). If this is repeatedly differentiated, and at each step the element $D^p v_j$ is eliminated by use of the equation (6. 1), the results are the formulas

$$D^i m^*(\eta_j) = \lambda^q D^i v_j + \lambda^{q+i-r} \sum_{\mu=0}^{p} \lambda^{1-\mu} \sigma^{(i)}_{\mu, r} D^{\mu-1} v_j, \quad i = 0, 1, 2, \dots . \tag{8.5}$$

We may write this also, with the use of the symbol $\delta_{i,j}$ to denote 1 when $j = i$ and 0 when $j \neq i$, in the form

$$D^{i-1} m^*(\eta_j) = \lambda^{q+i-l} \sum_{\mu=1}^{p} \left\{ \delta_{i,\mu} + \frac{\sigma^{(i-1)}_{\mu, r}}{\lambda^r} \right\} \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}}. \tag{8.6}$$

This shows, now, at once, that the determinant T can be factored, thus

$$T = \lambda^{pq} E W_p(v) \tag{8.7}$$

in which E is the determinant whose element in the i^{th} row and j^{th} column is indicated thus

$$E = \left| \delta_{i,j} + \frac{\sigma^{(i-1)}_{j, r}}{\lambda^r} \right|. \tag{8.8}$$

It is clear that E differs from 1 by terms of at least the degree r in $1/\lambda$. Since $W_p(v)$ and $W_q(y)$ are non-vanishing, it follows from (8. 3) and (8. 7) that the same is true of W_n.

9. The related equation.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0 . \tag{9. 1}$$

with

$$L^*(u) = \frac{1}{T}\begin{bmatrix} m^*(\eta_1) & - - - - - - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - - - - - - & Dm^*(\eta_p) & Dm^*(u) \\ - & - - - - - - & - & - \\ - & - - - - - - & - & - \\ D^{p-1}m^*(\eta_1) & - - - - - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - - - - - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix} \cdot \tag{9. 2}$$

T being the determinant given in (8. 4). This is clearly a differential equation of the n^{th} order in u, for which each one of the functions $y_j(z, \lambda)$ and $\eta_i(z, \lambda)$ is a solution. For if η_i is substituted for u two of the columns of the determinant (9. 2) are the same, and if u is replaced y_j every element of the last column vanishes. Because the n solutions thus produced are linearly independent the solutions of the equation (9. 1) are completely known.

The co-factor of the element $l^*(m(u))$ in the formula (9. 2) is the determinant T. The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{\nu=1}^{p} \frac{T_\nu}{T} D^{p-\nu} m^*(u) , \tag{9. 3}$$

where T_v is the determinant that is obtainable from the formula (8. 4) by replacing its elements $D^{p-\nu} m^*(\eta_j)$ by $l^*(m^*(\eta_j))$.

From the formula (8. 5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{\nu=1}^{p} \frac{\tau_\nu(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \tag{9. 4}$$

with

$$\tau_\nu(z, \lambda) = \sum_{k=0}^{p} \bar{\beta}_k(z, \lambda)\, \sigma_{\nu, r}^{(p-k)}(z, \lambda) . \tag{9. 5}$$

The replacements which change T to T_ν are thus seen to be ones which replace

$$\lambda^{n-\nu}\left\{\delta_{p-\nu,j}+\frac{\sigma_{j,r}^{(p-\nu)}}{\lambda^r}\right\} \quad \text{by} \quad \lambda^n\frac{\tau_\nu}{\lambda^r}.$$

It follows that

$$\frac{T_\nu}{T}=\lambda^\nu\frac{\theta_\nu(z,\lambda)}{\lambda^r},$$

with some function $\theta_\nu(z,\lambda)$ which is bounded over the z and λ domains. This gives to the relation (9. 3) the form

$$L^*(u)=l^*(m^*(u))-\frac{1}{\lambda^r}\sum_{\nu=1}^{p}\lambda^\nu\theta_\nu D^{p-\nu}m^*(u). \quad (9.7)$$

With the substitution of the expression for $D^{p-\nu}m^*(u)$, as it may be obtained from (4. 3) by writing $\bar{\gamma}_{i-s}$ in the place of γ_{i-s}, it is found that

$$L^*(u)=l^*(m^*(u))-\frac{1}{\lambda^r}\sum_{j=1}^{n}\lambda^j\omega_j(z,\lambda)D^{n-j}u, \quad (9.8)$$

with

$$\omega_j(z,\lambda)=\sum_{\nu=1}^{p}\sum_{s=0}^{p}\lambda^{-s}\binom{p-\nu}{s}\theta_\nu D^s\bar{\gamma}_{\mu-\nu-s}.$$

A comparison of this with the earlier result (6. 6) shows that

$$L^*(u)=L(u)-\frac{1}{\lambda^r}\sum_{j=1}^{n}\lambda^j\{\varepsilon_j(z,\lambda)+\omega_j(z,\lambda)\}D^{n-j}u. \quad (9.9)$$

The equation (9. 1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2. 1) only by terms that are of at least the r^{th} degree in $1/\lambda$. It is, therefore, by definition, a related equation.

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